Performance of Extremal Codes

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Outline

1. What codes do perform better?
2. What codes are extremal?
3. How to study performance of extremal codes?
4. Concluding remarks

Introduction

- Linear $[n, k, d]$ code C is used for data transmission

$$A(x, y) = \sum_{i=1}^{n} A_i x^{n-i} y^i,$$

A_i is the number of codewords of C of weight i

- Symbol error probability is p
- Bounded distance decoding is used
- Up to $t \leq \frac{d-1}{2}$ errors are corrected

What do we call “performance”?

Probability of erroneous decoding from the transmitter and receiver points of view:

$$P_{tr}(C, t, p) = P\left(Y \in \bigcup_{c \neq c' \in C} B_t(c') \;\middle|\; X = c \right),$$

$$P_{rv}(C, t, p) = P\left(X \in C \setminus \{c\} \mid Y \in B_t(c) \right),$$

with the random variables

- X – “the sent codeword”,
- Y – “the received vector”.

What codes perform better?

Theorem (Faldum, Lafuente, Ochoa, Willems, '06)

Let C and C' be $[n, k, d]$ codes with weight enumerators $A(x, y)$ and $A'(x, y)$ respectively. If p is small enough, then the following conditions are equivalent:

(a) $\mathrm{P}_{tr}(C, t, p) \leq \mathrm{P}_{tr}(C', t, p)$,

(b) $\mathrm{P}_{rv}(C, t, p) \leq \mathrm{P}_{rv}(C', t, p)$,

(c) $A(1, y) \preceq A'(1, y)$, *where "$\preceq$" means lexicographical ordering.*

Remark

"$\prec$" means $A_d < A'_d$,
or $A_d = A'_d$ and $A_{d+1} < A'_{d+1}$,
or ...

Self-dual codes

- $C^\perp = \{u \mid u \cdot v = 0 \text{ for all } v \in C\}$ is the dual code
- If $C = C^\perp$ the code is self-dual ($n = 2k$)
- Two types of self-dual codes:
  Type I (singly-even): all weights are even
  Type II (doubly-even): all weights are a multiple of 4

Theorem (Gleason '70)

Weight enumerator $A(x, y)$ of a self-dual code is a polynomial in two invariants f and g, that are

- *for Type I codes:* $$f = x^2 + y^2, \quad g = x^2y^2\left(x^2 - y^2\right)^2,$$
- *for Type II codes:* $$f = x^8 + 14x^4y^4 + y^8, \quad g = x^4y^4\left(x^4 - y^4\right)^4.$$

Self-dual codes

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Corollary

- *for Type II codes:* $$f = x^8 + 14x^4y^4 + y^8,$$ $$g = x^4y^4\left(x^4 - y^4\right)^4.$$

Length of a Type II code is a multiple of 8

$$n = 24m + 8i, \quad i = 0, 1 \text{ or } 2$$

Extremal doubly-even codes

Corollary (Mallows, Sloane '73)

$$\textit{for Type I codes} \qquad d \le 2\left\lfloor \frac{n}{8} \right\rfloor + 2,$$

$$\textit{for Type II codes} \qquad d \le 4\left\lfloor \frac{n}{24} \right\rfloor + 4.$$

- If "=" codes are called extremal
  Weight enumerator is unique
- Zhang '99: no extremal Type II codes for $n > 3952$
- Extremal Type II codes are known only up to $n = 136$
- The bound for Type I codes is NOT tight

Shadows of self-dual codes

- C is a Type I $[n, n/2, d]$-code
  C_0 is a doubly-even subcode; $C_2 := C \setminus C_0$
- Shadow $S = S(C)$ consists of all u, such that:

$$\begin{array}{ll} u \cdot v = 1 & \text{for all } \; v \in C_0 \\ u \cdot v = 0 & \text{for all } \; v \in C_2 \end{array}$$

- S is a non-linear code with weight enumerator $S(x, y)$
- $S(x, y) = A\left(\frac{x+y}{\sqrt{2}}, i\frac{x-y}{\sqrt{2}}\right)$
- If $8 \mid n$ then all weights in S are divisible by 4

Extremal singly-even codes

- C is a Type I $[n, n/2, d]$-code
- MALLOWS, SLOANE '73: $d \leq 2 \left\lfloor \frac{n}{8} \right\rfloor + 2$ (not tight)

Theorem (RAINS '98)

$$d \leq 4 \left\lfloor \frac{n}{24} \right\rfloor + 4, \quad n \not\equiv 22 \bmod 24,$$
$$d \leq 4 \left\lfloor \frac{n}{24} \right\rfloor + 6, \quad n \equiv 22 \bmod 24.$$

If $n = 24m$ Type I codes do not reach the bound

- If $n \equiv 8$ or $16 \bmod 24$, both Type I and Type II extremal codes have the same minimal distance

Comparing self-dual and non self-dual codes

- C is a self-dual extremal code of Type II
- C' is a non self-dual code with the same parameters

0	...	d	$d+1$	$d+2$	$d+3$	$d+4$	$d+5$	...	$\sum$
1	$0\dots0$	A_d	0	0	0	$*$	0	...	2^k
1	$0\dots0$	A'_d	$*$	$*$	$*$	$*$	$*$	...	2^k

- $A'(x,y) \prec A(x,y)$ is conjectured,
  i.e. C' is expected to perform better than C

Counterexample (Cheng, Sloane '89)

- C and C' are $[32,16,8]$-codes
- $A_d = 620 < 681 = A'_d$
- Conjecture is not correct

Comparing self-dual codes for small lengths

$$n = 24m + 8 \text{ or } 24m + 16$$

n	d	A_d for Type II	A_d for Type I
32	8	620	364
40	8	285	$125 + 16\beta$ ($\beta < 10$, $10 \leq \beta \leq 26$) (two known codes with $A_d = 285$)
56	12	8 190	$\leq 4\,862$
64	12	2 976	$1\,312 + 16\beta$ ($\beta < 104$, $104 \leq \beta \leq 284$)
80	16	97 565	$\leq 66\,845$
104	20	1 136 150	$\leq 739\,046$

Type I codes with unique weight enumerator

- s – minimum weight of the shadow S
- BACHOC, GABORIT '04: $2d + s \leq \frac{n}{2} + 4$
  If "=" the code is s-extremal
  A_d is known for s-extremal codes
- If s is smallest possible
  the code is with minimal shadow

If $n = 24m + 8$:

$s = 4m$ for s-extremal codes
$s = 4$ for codes with minimal shadow

Best extremal codes of Type I

C is a code of Type I with shadow S
s – minimum weight of the shadow

$$A^{(s)}(1,y) = 1 + A_d^{(s)} y^d + A_{d+2}^{(s)} y^{d+2} + \cdots + y^n$$

$$A_d^{(4m)} < A_d^{(s)} \quad \text{for all } 4 \le s < 4m \qquad (\text{Bouyuklieva})$$

Moreover, we can express $A_d^{(4)}$ through $A_d^{(4m)}$.

Comparing Type I and Type II extremal codes

$$n = 24m + 8$$

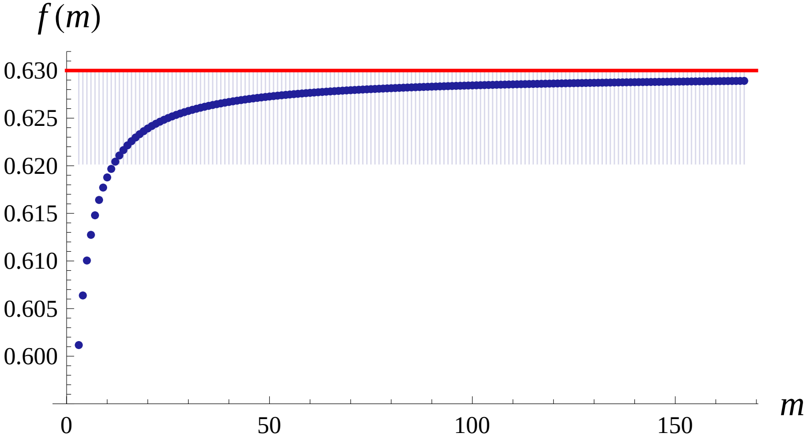
- C – Type II extremal code
- C' – Type I extremal code with min shadow

$$f(m) = \frac{A'_d}{A_d} < 1$$

- C' performs better than C

$\Rightarrow$ s-extremal codes are better than Type II codes

Behaviour of $f(m)$



Concluding remarks

- $n = 24m + 8$
- A lot of different weight enumerators for Type I codes
- $A_d^{(4m)} < \ldots < A_d^{(s_i)} < \ldots < A_d^{(4)} < A_d^{(s_j)} < \ldots < A_d^{(s_k)}$
- For the codes in the tail the problem is not solved

Thank you!