Leibniz
Universität
Hannover

Extremal Codes with 2-transitive Groups

Anton Malevich

joint work with Wolfgang Willems

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Self-Dual Type II Codes

- Linear code C is a subspace of $\mathbb{F}^n$, $\mathbb{F} = \mathbb{F}_2$,
  $c \in C$ is a codeword
- The dual code
  $$C^\perp = \{v \mid \langle u, v \rangle = 0 \text{ for all } u \in C\}$$
  If $C = C^\perp$ the code is self-dual
- For a self-dual code dim $= {}^n\!/_2$
- Weight of c is the number of 1's
- Self-dual code is of Type II
  if all weights are divisible by 4

Example: Hamming Code

$$\begin{array}{cc} \begin{matrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

- C is a subspace of $\mathbb{F}^8$ spanned by rows
- Self-dual: $\langle u, v \rangle = 0$ for all $u, v \in C$
- Type II: all weights are divisible by 4
- Minimum distance: $d = 4$

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- C is a subspace of $\mathbb{F}^8$ spanned by rows
- Self-dual: $\langle u, v \rangle = 0$ for all $u, v \in C$
  - $\langle c_1, c_2 \rangle = 0 + 0 + 1 + 0 + 0 + 0 + 0 + 1 = 0$
  - $\langle c_i, c_j \rangle = 0$ for all $i, j \in \{1, 2, 3, 4\}$
- Type II: all weights are divisible by 4
- Minimum distance: $d = 4$

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- Self-dual: $\langle u, v \rangle = 0$ for all $u, v \in C$
- Type II: all weights are divisible by 4
  - $\mathrm{wt}(c_i) =$ # of 1's $= 4$
  - $\mathrm{wt}(c) = 0, 4$ or 8 for $c \in C$
- Minimum distance: $d = 4$

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- Self-dual: $\langle u, v \rangle = 0$ for all $u, v \in C$
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- Minimum distance: $d = 4$
  - $d = \min\{\mathrm{wt}\, c \,|\, c \in C,\ c \neq 0\}$

Extremal Type II Codes

- Bound on d: $d \leq 4 \left\lfloor \frac{n}{24} \right\rfloor + 4$,

  If "=" then the code is extremal

- For extremal codes $n \leq 3928$
- Length of a Type II code is a multiple of 8
- Extremal codes only constructed for $n =$

  8, 16, 24, 32, 40, 48, 56, 64, 80, 88, 104, 112, 136

- Our concern: $136 < \;\overset{?}{\ldots}\; \leq 3928$

Automorphism Group

- $\mathrm{Aut}(C) = \{\sigma \in S_n \mid c\sigma \in C \text{ for all } c \in C\}$

Example: Extended cyclic code

$\sigma = (1\,2\,3\,4\,5\,6\,7)$ – cyclic shift, (8) is fixed

$$\begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \hline \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \end{array} \overset{\sigma}{\mapsto} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \hline \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

- C is an $\mathbb{F}G$-module of dim $^n/_2$
- $G \leq \mathrm{Aut}(C)$ helps construct a code

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- C is an $\mathbb{F}G$-module of dim $n/2$
- $G \leq \mathrm{Aut}(C)$ helps construct a code

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Extremal Type II Codes (cont.)

- Extremal codes only known for $n =$
  8, 16, 24, 32, 40, 48, 56, 64, 80, 88, 104, 112, 136
- Common approach: one length n at a time
  1. Assume $G \leq \mathrm{Aut}(C)$ for some G
  2. Construct extremal C
     (or prove nonexistence under the assumption)
- SLOANE'73: $n = 72$? Still open
  - …, BORELLO'13: $|\mathrm{Aut}(C)| \leq 5$
  - Only 6 possibilities for $\mathrm{Aut}(C)$
- HARADA'08: $n = 112$

Extremal Type II Codes (cont.)

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- Common approach: one length n at a time
  1. Assume $G \leq \mathrm{Aut}(C)$ for some G
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     (or prove nonexistence under the assumption)
- Our approach: all lengths $n \leq 3928$
  - Families of codes: QR, QDC
  - Automorphisms of prime order $p \geq {}^{n}\!/_{2}$
  - 2-transitive $\mathrm{Aut}(C)$

Quadratic Residue Codes

- Exist for $n = p + 1$,
  p prime, 2 a square in $\mathbb{F}_p$

Example: $n = 8$, $p = 7$

$1, 2$ and 4 are
the squares in $\mathbb{F}_7^\times$

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Theorem (Bouyuklieva, M., Willems’10)
The only extremal QR codes are of lengths

$$n = 8, 24, 32, 48, 80 \text{ and } 104$$

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Sketch of the Proof

- Task: find a codeword of weight $< 4\left\lfloor \frac{n}{24} \right\rfloor + 4$ in every QR code for $n \leq 3928$
- How? Search in a subcode

  $$C^H = \{\text{codewords fixed by all } \sigma \in H\},$$

  where $H \leq \mathrm{Aut}(C)$ suitable
- How to find suitable H? (heuristic)
  - $|H|$ large $\Leftrightarrow$ $\left|C^H\right|$ small
  - $\left|C^H\right|$ depends on structure of H
  - $5 \leq |H| \leq 30$ works for large n

2-Transitive Automorphism Groups

Known extremal codes with 2-tr. Aut(C)

- Quadratic Residue codes of lengths:
  $8, 24, 32, 48, 80, 104$
- Reed-Muller code of length 32

Theorem (M., WILLEMS'12 + CHIGIRA ET AL.'13)
There are no other such codes.

Example: Hamming Code

1. Aut(C) is transitive $=$ for any $i, j \in \{1, \ldots, n\}$ there exists $\tau \in \mathrm{Aut}(C)$ with $\tau(i) = j$

$i = 1, j = 8$: $\tau_1 = (1\,8)(2\,4)(3\,7)(5\,6) \in \mathrm{Aut}(C)$

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \overset{\tau_1}{\mapsto} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

- $E = \{\mathrm{id}, \tau_1, \ldots, \tau_7\}$ elementary abelian
  $|E| = \deg E = n$, E is transitive

Example: Hamming Code

2. $\mathrm{Aut}(C)$ is 2-transitive $=$ transitive and for any $i, j \in \{1, \dots, n-1\}$ there exists $\sigma \in \mathrm{Aut}(C)$ with $\sigma(i) = j$ and $\sigma(n) = n$

$i = 1, j = 2$: $\sigma = (1\,2\,3\,4\,5\,6\,7) \in \mathrm{Aut}(C)$

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \overset{\sigma}{\mapsto} \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

- $E \rtimes \langle \sigma \rangle = \mathrm{AGL}(1, 2^3)$ is 2-transitive
- $\mathrm{AGL}(1, 2^m) \leq \mathrm{Aut}(C) \Rightarrow C$ affine invariant

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The Method

- $G = \mathrm{Aut}(C)$ is 2-transitive

1. Use the structure of G
    - The socle of G is simple or elementary abelian
    - Degree of G = length of $C \leq 3928$
    - $\Rightarrow$ Only few possibilities for G
2. Find all $\mathbb{F}\,G$-modules of dim $^n/_2$
3. Find modules that are self-dual as codes
4. Check if the codes are extremal
    - Use subgroups of G

Simple Socle

Socle	$n^\dagger$	dim ${}^{n}/_{2}$ mod.	extremal
M_{24}	24	Golay code	yes
HS	176	none	
$PSU(3,7)$	344	none	
$PSL(2,7^3)$	344	GQR code	no
$PSL(m,q)$	4 pos.	none	
$PSp(2m,2)$	6 pos.	none	
$PSL(2,p)$	$p+1$	QR codes	$n \leq 104^*$
A_n	n	none	

† $8 \mid n$, $n \leq 3952$

* QR codes Theorem

Simple Socle: Case A_n

- $\mathrm{A}_n = \{\sigma \in S_n \mid \mathrm{sgn}(\sigma) = 1\}$

Proposition

There are no extremal codes with $\mathrm{Aut}(C) = \mathrm{A}_n$.

$$C \ni c = \left(\begin{array}{ccccccccc} & & i & & j & & k & & \\ \hline * & \overset{*}{\ldots} & 0 & \overset{*}{\ldots} & 1 & \overset{*}{\ldots} & 1 & \overset{*}{\ldots} & * \end{array}\right)$$

- $\sigma = (i, j, k) \in \mathrm{A}_n$, since $\mathrm{sgn}(\sigma) = 1$.

$$c + c\sigma = \left(\begin{array}{ccccccccc} & & i & & j & & k & & \\ \hline 0 & \overset{0}{\ldots} & 1 & \overset{0}{\ldots} & 1 & \overset{0}{\ldots} & 0 & \overset{0}{\ldots} & 0 \end{array}\right)$$

- $\mathrm{wt}(c + c\sigma) = 2 \Rightarrow C$ not extremal

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$PSL(m, q)$	4 pos.	none	
$PSp(2m, 2)$	6 pos.	none	
$PSL(2, p)$	$p + 1$	QR codes	$n \leq 104^*$
A_n	n	none	

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* QR codes Theorem

Elementary Abelian Socle E

- $|E| = n = 2^m$, $m \leq 11$ (since $n \leq 3928$)
- $G \leq \mathrm{AGL}(m, 2)$ 2-transitive

$\Rightarrow$ $G \cong E \rtimes H$, $H \leq \mathrm{GL}(m, 2)$ transitive

- Two cases:

1. C affine invariant
   - H contains cyclic shift σ of length $(n - 1)$
2. C not affine invariant

Affine Invariant Codes

- $\mathrm{AGL}(1, 2^m) \leq G \leq \mathrm{AGL}(m, 2)$
- $n = 2^m$, m is odd
- CHARPIN, LEVY-DIT-VEHEL'94:
  A method to construct all aff. inv. codes

m	n	Num of codes	extremal
5	32	1	yes
7	128	3	none
9	512	70	none
11	2048	515 617	none

Other Cases

- $G \cong E \rtimes H$, $H \leq \mathrm{GL}(m, 2)$ is transitive
  - H does not contain cyclic shift σ

- $n = 2^m$, $m = 4, 6, 8, 9$ or 10

- Possibilities for H:
  - $\mathrm{PSL}(k, 2^r) \leq H$, $m = kr \quad k, r \geq 2$
  - $\mathrm{PSp}(k, 2^r) \leq H$, $m = kr$, k even
  - Sporadic examples for $m = 4, 6$

Other Cases (cont.)

- $n = 2^m$, $m = 4, 6, 8, 9$ or 10
- $G = E \rtimes \mathrm{PSL}(k, 2^r)$, $m = kr$
- For $m < 9$: no self-dual codes
- Only for $m = 9$: 3 codes, not extremal
- $m = 10$: Too many $\mathbb{F}\,G$-modules of dim $n/2$
  - CHIGIRA ET AL.'13: for $m = 2sr$ even
    no self-dual codes with $G \leq E \rtimes \mathrm{PSL}(2s, 2^r)$

Summary

- Extremal codes with 2-tr. Aut(C) are known
  - QR codes of length 8, 24, 32, 48, 80 or 104
  - Reed-Muller code of length 32

$\Rightarrow$ If new extremal codes exist,
then they have “little” structure

- Open problems
  - Classify self-dual codes with 2-tr. Aut(C)
  - Reduce the bound $n \leq 3928$ for extremal codes

Thank you for your attention!