Übungen zur Stochastik II **WS 2014/2015**

Blatt 1

Problem 1.1 a) Let X, Y be real-valued random variables with $\mathbb{E}[|Y|] < \infty$ and assume that the joint law $\mathscr{L}(X,Y)$ has density $\varphi_{X,Y}$ w.r.t. Lebesgue measure $\lambda \otimes \lambda$ on $\mathbb{R}^2$, i.e. $\mathbb{P}\big((X,Y) \in B)\big) = \int_B \varphi_{X,Y}(x,y)\, \lambda \otimes \lambda(dxdy)$ for any $B \in \mathcal{B}(\mathbb{R}^2)$. Let $\varphi_X(x) := \int_{\mathbb{R}} \varphi_{X,Y}(x,y)\, \lambda(dy)$ be the (marginal) density of X and put

$$\psi(x) := \begin{cases} \displaystyle\int_R y \frac{\varphi_{X,Y}(x,y)}{\varphi_X(x)}\, \lambda(dy) & \text{if } \varphi_X(x) > 0, \\ 0 & \text{else.} \end{cases}$$

Show that the random variable $\psi(X)$ is a version of $\mathbb{E}[Y|X]$.

b) Let U, V be independent and uniformly distributed on $[0,1]$.

i) Compute $\mathbb{E}\left[U \mid \frac{U}{V}\right]$.

ii) Compute $\mathbb{E}\left[U \mid U - V\right]$.

iii)* Discuss in view of i) and ii) how the term $\mathbb{E}\left[U \mid U = V\right]$ could or could not be interpreted.

[*Hint:* Use the transformation formula to compute the joint density of $(U, U/V)$ and of $(U, U-V)$, then apply a).]

Problem 1.2 a) Show that if $\mathcal{F} \subseteq \mathcal{G}$ and $\mathbb{E}[X^2] < \infty$, then

$$\mathbb{E}\big[\big(X - \mathbb{E}[X|\mathcal{G}]\big)^2\big] + \mathbb{E}\big[\big(\mathbb{E}[X|\mathcal{G}] - \mathbb{E}[X|\mathcal{F}]\big)^2\big] = \mathbb{E}\big[\big(X - \mathbb{E}[X|\mathcal{F}]\big)^2\big].$$

b) Show that if X and Y are random variables with $\mathbb{E}[X|\mathcal{G}] = Y$ and $\mathbb{E}[X^2] = \mathbb{E}[Y^2] < \infty$, then $X = Y$ a.s.

Problem 1.3 Let $(\mathcal{F}_n)_{n \in \mathbb{N}_0}$ be a filtration and $(X_n)_{n \in \mathbb{N}_0}$ an $(\mathcal{F}_n)_{n \in \mathbb{N}_0}$-martingale. If $(X_n)_n$ is predictable, then $X_n = X_0$ for all $n \in \mathbb{N}$ almost surely.

Problem 1.4 a) Let $(X_n)_n$, $(Y_n)_n$ be submartingales (with respect to the same filtration), then $(X_n \vee Y_n)_n$ is a submartingale, too.

b) Let $(X_n)_n$ be a martingale, $\varphi : \mathbb{R} \to \mathbb{R}$ convex with $\mathbb{E}[\varphi(X_n)^+] < \infty$ for all n. Then $(\varphi(X_n))_n$ is a submartingale.

c) If in b) we replace "martingale" by "submartingale" and assume in addition that φ is non-decreasing then the conclusion from b) holds.

d) Show that in c) the requirement that φ is non-decreasing cannot be dispensed with in general.

Problem 1.5 Let $V_1, V_2, \ldots$ be independent and identically distributed non-negative random variables and put $X_n := \prod_{i=1}^n V_i$ (we interpret the empty product as $X_0 := 1$).

a) Under what conditions is $(X_n)_n$ a supermartingale? When is it a martingale?

b) If $(X_n)_n$ is a supermartingale and $\Pr(V_1 = 1) < 1$ then $X_n \to 0$ a.s. as $n \to \infty$.

Abgabe der Aufgaben: 4.11.2014 (Es findet keine Bewertung statt.)