Übungen zur Stochastik II **WS 2014/2015**

Blatt 2

Problem 2.1 A monkey types one letter per second, with each letter drawn independently and uniformly distributed from the 26 possible letters $\{\mathtt{a}, \mathtt{b}, \mathtt{c}, \ldots, \mathtt{z}\}$. Check that the expected time until the pattern `abracadabra` appears for the first time is $26^{11}+26^4+26$ ($= 3{,}670{,}344{,}487{,}444{,}778 \doteq 3.67 \times 10^{15}$) seconds.

[Hint: Consider a fair "casino" where one can bet in each round on a new random letter. A set of players successively enters the game and, starting from 1€, bet on the occurence of the required pattern, always betting their current winnings until either the pattern occurs or they go broke. Check that their accumulated gains process is a martingale and use a suitable stopping argument.]

Problem 2.2 Find an $\mathscr{L}^2$-martingale (M_n) with $M_n \to M_\infty$ a.s. for a finite r.v. M_∞ but $\langle M \rangle_n \to \infty$ a.s.

[Hint: Consider for example a sequence with independent but not stationary increments.]

Problem 2.3 a) Let (M_n) be a martingale with $|M_n - M_{n-1}| \leq c$ for some fixed $c < \infty$. Check that the two disjoint events

$$C := \{M_n \text{ converges to a finite limit}\},\ F := \{\limsup M_n = +\infty, \liminf M_n = -\infty\}$$

satisfy $\mathbb{P}(C \cup F) = 1$.

[Hint: To show that $\mathbb{P}(\{\liminf M_n > -\infty\} \cap C^c) = 0$ check that the stopped process $(M_{n \wedge T_K} - M_0)_n$ with $T_K := \inf\{m \in \mathbb{N}_0 : M_m \leq -K\}$ is a martingale with $M_{n\wedge T_K} \geq -c - K$ for all n.]

b) (Conditional Borel-Cantelli lemma) Let $A_1, A_2, \ldots$ be event with $A_n \in \mathcal{F}_n$. Show that

$$\left\{\sum_{n=1}^\infty \mathbb{P}(A_n | \mathcal{F}_{n-1}) = \infty\right\} = \limsup_{n\to\infty} A_n \quad \text{a.s.}$$

[*Hint*: Consider the martingale $X_n = \sum_{i=1}^n \big(\mathbf{1}_{A_i} - \mathbb{P}(A_i|\mathcal{F}_{i-1})\big)$.]

Problem 2.4 Let $p \in [0,1]$, consider a stochastic process $(X_n)_{n\in\mathbb{N}_0}$ with $X_0 = x_0 \in [0,1]$ and the following dynamics: For $n \in \mathbb{N}_0$, conditional on $X_0, X_1, \ldots, X_n$, we have

$$X_{n+1} = \begin{cases} 1 - p + pX_n & \text{with probability } X_n \\ pX_n & \text{with probability } 1 - X_n \end{cases}$$

Check that (X_n) is a martingale and converges a.s. What is the law of its limit?

[Hint: If X_n converges to X_∞, show that the event $\{a < X_\infty < b\}$ must have probability 0 for any $0 < a < b < 1$.]

Problem 2.5 Let $N \in \mathbb{N}$, $p \in (0,1)$ and consider a probability space on which there exist i.i.d. random variables $D_1, \ldots, D_N$ which take the values $+1$, resp. -1, with probability p, resp., $1-p$. Let $\mathscr{F}_n := \sigma(D_1, \ldots, D_n)$ (with $\mathscr{F}_0 := \{\emptyset, \Omega\}$).

(A "canonical" choice is $(\Omega, \mathscr{F}, \mathbb{P}_p)$ with $\Omega = \{-1,+1\}^N$, $\mathscr{F} = 2^\Omega$, $\mathbb{P}_p(\{(\omega_1, \ldots, \omega_N)\}) = p^{\#\{i:\omega_i=+1\}}(1-p)^{\#\{i:\omega_i=-1\}}$ and $D_i : \Omega \to \{-1,+1\}$, $D_i((\omega_1, \ldots, \omega_N)) = \omega_i$.)

a) (A discrete martingale representation theorem) Check that $Z_0 := 0$, $Z_n := \sum_{i=1}^{n}(D_i - 2p + 1)$, $n = 1, \dots, N$ is a martingale (w.r.t. $(\mathscr{F}_n)$). Let $(M_n)_{n=0,\dots,N}$ be a martingale. Show that there exists a previsible process (H_n) such that $M_n = M_0 + (H \bullet Z)_n$.

b) (A simple market model, the Cox-Ross-Rubinstein model) Let $0 < a < 1 + r < b$, put $B_n := (1 + r)^n$, with a fixed value $S_0 > 0$ define

$$S_n := S_{n-1}\big(\tfrac{b+a}{2} + \tfrac{b-a}{2} D_n\big), \quad n = 1, \dots, N.$$

We think of B_n as the value of a risk-free investment, a "bond", and of S_n as the value of a risky investment, a "stock", at time n. Note that $R_n := S_n/S_{n-1}$, the "return" over the time period $[n - 1, n)$, takes two possible values a and b.

A pair of previsible processes $(V_n)_n$, $(W_n)_n$ is called a *self-financing strategy* if

$$V_{n-1}B_{n-1} + W_{n-1}S_{n-1} = V_n B_{n-1} + W_n S_{n-1}, \quad n = 1, \dots, N \tag{1}$$

holds. Then $X_n := V_n B_n + W_n S_n$, $n \in 1, \dots, N$ (with $X_0 := V_1 B_0 + W_1 S_0$) is the value at time n of a portfolio where a trader holds V_n units of the bond and W_n units of the stock over the interval $(n - 1, n)$ and possibly re-arranges her investments between such intervals. Check that $X_n = X_0 + (V \bullet B)_n + (W \bullet S)_n$ and if $p = p^* := (1 + r - a)/(b - a)$, $Y_n := X_n/(1 + r)^n$ is a martingale.

c) (The price of a European call option in the CRR model) Let $K > 0$, $C := (S_N - K)^+$ the value at time N of a contingent claim whose holder has the right to buy one unit of the stock at time N for the price K if she chooses (a European call with strike price K).

Describe a self-financing stategy with initial value $X_0 = \mathbb{E}_{p^*}[C/(1+r)^N]$ (where $p^* = (1 + r - a)/(b - a)$) and terminal value $X_n = C$.

[Hint: Consider $Y_n = \mathbb{E}_{p^*}[C/(1 + r)^N | \mathscr{F}_n]$, use a).]

Abgabe der Aufgaben: 18.11.2014 (Es findet keine Bewertung statt.)