Übungen zur Stochastik II WS 2014/2015

Blatt 3

Problem 3.1 a) Let $n \in \mathbb{N}$, $n \geq 2$, assume that $X_1, \dots, X_n$ are exchangeable real-valued random variables with $\mathbb{E}[X_1^2] < \infty$. Then we have

$$\mathrm{Cov}(X_1, X_2) \geq -\frac{1}{n-1}\mathrm{Var}(X_1). \tag{1}$$

Find an example where equality holds in (1).
[*Hint:* Consider the variance of $X_1 + \cdots + X_n$.]
b) Let $X_1, X_2, \dots$ be exchangeable real-valued random variables with $\mathbb{E}[X_1^2] < \infty$. Show that $\mathrm{Cov}(X_1, X_2) \geq 0$.
c) Give an example of a finite exchangeable family $(X_1, X_2, \dots, X_n)$ which can not be extended to an infinite exchangeable family $X_1, X_2, \dots$.

Problem 3.2 Let $X_1, X_2, \dots$ i.i.d. with values in $\mathbb{Z}$, put $S_0 := 0$, $S_n := X_1 + \cdots + X_n$, $R_n := \{S_n = 0\}$ for $n \in \mathbb{N}$.
Show that $\mathbb{P}\big(\limsup_{n\to\infty} R_n\big) \in \{0, 1\}$.

Problem 3.3 ("Ballot theorem") Let $X_1, X_2, \dots$ be i.i.d. with values in $\mathbb{N}_0$, put $S_0 := 0$, $S_n := X_1 + \cdots + X_n$ for $n \in \mathbb{N}$. Show that

$$\mathbb{P}\big(S_j < j \text{ for all } j = 1, \dots, n \,\big|\, S_n\big) \geq (1 - S_n/n)^+. \tag{2}$$

Furthermore, if $\mathbb{P}(X_1 \in \{0,1,2\}) = 1$ then equality holds in (2).
[*Hint:* Check that $Y_k := S_{-k}/(-k)$, $k = -n, -n+1, \dots, 1$ is a martingale w.r.t. the filtration $\mathscr{F}_k = \sigma(S_{-k}, S_{-k+1}, \dots, S_n)$, $T := \inf\big\{k \in \{-n, -n+1, \dots, -1\} : Y_k \geq 1\big\} \wedge (-1)$ is a stopping time, with $G := \{S_j < j \text{ for all } j = 1, \dots, n\}$ we have $1_{G^c} \leq Y_T$, then use optional sampling.

Furthermore, if $\mathbb{P}(X_1 \in \{0,1,2\}) = 1$ check that with $\varrho := \max\{0 \leq j \leq n : S_j - j \geq 0\}$, $\{S_n - n < 0\} \subset \{S_\varrho - \varrho = 0\}$ a.s. and conclude that $\{S_n - n < 0\} \cap G^c \subset \{Y_T = 1\}$ a.s.]

Problem 3.4 (On the limit composition in Pólya's urn) Recall Pólya's urn model: Initially, there are, say, M black and $N - M$ white balls in the urn ($M, N \in \mathbb{N}$, $N \geq M$). In each draw, we pick at random one of the balls currently in the urn and put it back together with a new ball of the same colour. (Apart from the colour, the balls are identical.) Then

$$X_n := 1_{\{\text{ball in } n\text{-th draw is black}\}}, \; n = 1, 2, \dots$$

are exchangeable and the asymptotic fraction of black balls

$$Z := \lim_{n\to\infty} \frac{M + X_1 + \cdots + X_n}{N + n} \quad \text{exists a.s.}$$

a) Compute $\mathbb{E}[Z^k]$ for $k \in \mathbb{N}$.
[*Hint:* Consider the joint law of $X_1, \dots, X_k$ conditional on the terminal (or the exchangeable) σ-algebra.]
b) The beta distribution $\beta_{a,b}$ with parameters $a, b > 0$ has density

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} 1_{(0,1)}(x) x^{a-1}(1-x)^{b-1}.$$

For which choice of a, b (depending on M, N) do the moments of Z agree with those of $\beta_{a,b}$?

Abgabe der Aufgaben: 2.12.2014 (Es findet keine Bewertung statt.)