Quantum Mechanics and Quantum Field Theory of Neutrino Oscillations

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based on work done in collaboration with
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Outline

1. **Motivation**

2. Wave packet approaches to neutrino oscillations
   - QM: Neutrino wave packets
   - QFT: Feynman diagrams with external wave packets
   - The connection between wave packet and plane wave approaches

3. Entanglement in neutrino oscillations

4. Neutrino flavor eigenstates

5. Summary and conclusions

Outline

The "textbook approach" to neutrino oscillations

Diagonalization of the mass terms of the charged leptons and neutrinos gives

$$\mathcal{L} \supset -\frac{g}{\sqrt{2}} \left(\bar{e}_{\alpha L} \gamma^\mu U_{\alpha j} \nu_{jL} \right) W_\mu^- \; + \; \text{diag. mass terms} \; + \; h.c.$$

(flavour eigenstates: $\alpha = e, \mu, \tau$, mass eigenstates: $j = 1, 2, 3$)

Assume, at time $t = 0$ and location $\vec{x} = 0$, a flavour eigenstate

$$|\nu(0,0)\rangle = |\nu_\alpha\rangle = \sum_i U_{\alpha j}^* |\nu_j\rangle$$

is produced. At time t and position $\vec{x}$, it has evolved into

$$|\nu(t, \vec{x})\rangle = \sum_i U_{\alpha j}^* e^{-iE_j t + i\vec{p}_j \vec{x}} |\nu_i\rangle$$

Oscillation probability:

$$P(\nu_\alpha \rightarrow \nu_\beta) = \left| \langle \nu_\beta | \nu(t, \vec{x}) \rangle \right|^2 = \sum_{j,k} U_{\alpha j}^* U_{\beta j} U_{\alpha k} U_{\beta k}^* e^{-i(E_j - E_k)t + i(\vec{p}_j - \vec{p}_k)\vec{x}}$$

Questions to think about

- What happens to energy-momentum conservation in neutrino oscillations?
- Under what conditions is coherence between different neutrino mass eigenstates lost?
- Is the standard oscillation formula universal, or are there situations where it needs to be modified?
- Why does the standard oscillation formula work so well, even though its derivation is plagued by inconsistencies?
- Are the neutrino and its interaction partners in an entangled state? If so, what are the phenomenological consequences of entanglement?
- Are flavor eigenstates well-defined and useful objects in QFT?

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Neutrino wave packets

Treat neutrino as superposition of mass eigenstate wave packets:

$$|\nu_\alpha S(t)\rangle = \int d^3 p \sum_j U_{\alpha j}^* f_{jS}(\vec{p}) \, e^{-iE_j(\vec{p})(t-t_S)} |\nu_j, \vec{p}\rangle$$

Advantage:

Heisenberg uncertainties and localization effects properly taken into account.

Disadvatage:

Assumptions on (or calculation of) shape functions $f_{jS}(\vec{p})$ needed.

Oscillation formula for neutrino wave packets

Detector projects $|\nu_{\alpha S}(t)\rangle$ onto a state $|\nu_{\beta D}\rangle$ with shape function $f_{kD}(\vec{p})$.

Oscillation probability (for Gaussian wave packets):

$$P_{ee} = \sum_{j,k} U_{\alpha j}^* U_{\beta j} U_{\alpha k} U_{\beta k}^* \exp\left[-2\pi i \frac{L}{L_{jk}^{\text{osc}}} - \left(\frac{L}{L_{jk}^{\text{coh}}}\right)^2 - 2\pi^2 \xi^2 \left(\frac{1}{2\sigma_p L_{jk}^{\text{osc}}}\right)^2 \right]$$

C. Giunti, C. W. Kim, U. W. Lee, Phys. Rev. **D44** (1991) 3635; C. Giunti, C. W. Kim, Phys. Lett. **B274** (1992) 87
K. Kiers, S. Nussinov, N. Weiss, Phys. Rev. **D53** (1996) 537, hep-ph/9506271
C. Giunti, C. W. Kim, Phys. Rev. **D58** (1998) 017301, hep-ph/9711363, C. Giunti, Found. Phys. Lett. **17** (2004) 103, hep-ph/0302026

with

$$L_{jk}^{\text{osc}} = 4\pi E/\Delta m_{jk}^2 \qquad \text{Oscillation length}$$
$$L_{jk}^{\text{coh}} = 2\sqrt{2}E^2/\sigma_p|\Delta m_{jk}^2| \qquad \text{Coherence length}$$

E	Energy that a massless neutrino would have
ξ	quantifies the deviation of E_i from E
σ_p	Effective wave packet width

The localization term

For $\xi \sim 1$, the term

$$\exp\left[-2\pi^2\xi^2\left(\frac{1}{2\sigma_p L_{jk}^{\mathrm{osc}}}\right)^2\right]$$

suppresses oscillations if wave packets are larger than the oscillation length. In that case, the phase difference between mass eigenstates varies appreciably during the detection process.

For $\xi \ll 1$ (equal energy limit, $E_i \simeq E_j$), this condition disappears (oscillations only in L, not in $T \rightarrow$ phase relation between mass eigenstates remains the same during the whole detection process).

The decoherence term

The term

$$\exp\left[- \left(\frac{L}{L_{jk}^{\text{coh}}} \right)^2 \right]$$

suppresses oscillations at long (astrophysical) baselines.

Interesting duality of interpretations

- Interpretation 1: (a single neutrino)
  Different mass eigenstates separate in space and time due to their different group velocities → loss of coherence
- Interpretation 2: (ensemble of neutrinos)
  At large baseline, oscillation maxima are so close in energy that the finite experimental resolution smears them out.

Reason:

A continuous flux of identical wave packets cannot be distinguished from an ensemble of plane waves with the same momentum distribution.
(Neutrino density matrices for the two ensembles are identical)

K. Kiers, S. Nussinov, N. Weiss, Phys. Rev. **D53** (1996) 537, hep-ph/9506271

The Feynman diagram of neutrino oscillations

Idea: Neutrino as intermediate line in macroscopic Feynman diagram; external particles described as wave packets or bound state wave functions
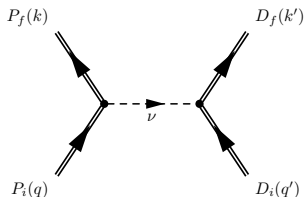
Advantages:

- Heisenberg uncertainties and localization effects properly taken into account.
- Neutrino properties (E_j, $\vec{p}_j$ of individual mass eigenstates, shape of wave function, ...) automatically determined by the formalism.
- Automatically yields properly normalized oscillation probabilities and event rates
- Includes dynamic modifications of mixing parameters due to different kinematics for different mass eigenstates (usually tiny, but may be important in sterile neutrino scenarios)

Oscillation probability from a Feynman diagram

Evaluation is tedious, but straightforward if external wave functions are Gaussian: ($\delta m_i^2 = m_i^2 - m_0^2$)

$$P_{\alpha\beta}(L) \propto \sum_{j,k} U_{\alpha j}^* U_{\alpha k} U_{\beta k}^* U_{\beta j} \exp\Bigg[-2\pi i \frac{L}{L_{jk}^{\text{osc}}} - \left(\frac{L}{L_{jk}^{\text{coh}}} \right)^2$$
$$-\frac{\pi^2}{2} \xi^2 \left(\frac{1}{\sigma_p L_{jk}^{\text{osc}}} \right)^2 - \frac{(\delta m_i^2 + \delta m_j^2)^2}{32\sigma_m^2 E^2} - \frac{(\delta m_i^2 - \delta m_j^2)^2}{32\sigma_m^2 E^2} \Bigg],$$

Four terms:

- Oscillation
- Decoherence (wave packet separation ↔ detector resolution effects)
- Localization (wave packet smaller than oscillation length or no oscillations in time)
- Approximate conservation of average energies/momenta
- Another localization effect: Suppression of oscillations if experimental resolutions allow determination of neutrino mass B. Kayser Phys. Rev. **D24** (1981) 110

A concrete example: Mössbauer neutrinos

Classical Mössbauer effect: *Recoilfree* emission and absorption of γ-rays from nuclei bound in a crystal lattice.

A similar effect should exist for neutrino emission/absorption in bound state β decay and induced electron capture processes.

W. M. Visscher, Phys. Rev. **116** (1959) 1581; W. P. Kells, J. P. Schiffer, Phys. Rev. **C28** (1983) 2162
R. S. Raghavan, hep-ph/0511191; R. S. Raghavan, hep-ph/0601079

Proposed experiment:

> Production: $^3\text{H} \rightarrow \,^3\text{He}^+ + \bar{\nu}_e + e^-(\text{bound})$
>
> Detection: $^3\text{He}^+ + e^-(\text{bound}) + \bar{\nu}_e \rightarrow \,^3\text{H}$

^{3}H and ^{3}He embedded in metal crystals (metal hydrides).

Physics opportunities:

- Neutrino oscillations on a laboratory scale: $E = 18.6$ keV, $L_{\text{atm}}^{\text{osc}} \sim 20$ m.
- Gravitational interactions of neutrinos
- Study of solid state effects with unprecedented precision

Mössbauer neutrinos (contd.)

Mössbauer neutrinos have very special properties: R. S. Raghavan, W. Potzel

- Neutrino receives *full* decay energy: $Q = 18.6$ keV
- Natural line width: $\gamma \sim 1.17 \times 10^{-24}$ eV
- Atucal line width: $\gamma \gtrsim 10^{-11}$ eV
  - ▶ Inhomogeneous broadening (Impurities, lattice defects)
    Each atom emits at slightly different, but constant, energy
    → ensemble of plane wave with different energies
  - ▶ Homogeneous broadening (Spin interactions)
    Energy levels of atoms fluctuate, but in the same way for all atoms
    → ensemble of wave packets with identical shapes

Remember:

A continuous flux of identical wave packets cannot be distinguished from an
ensemble of plane waves with the same momentum distribution.
(Neutrino density matrices for the two ensembles are identical)

K. Kiers, S. Nussinov, N. Weiss, Phys. Rev. **D53** (1996) 537, hep-ph/9506271

Transition amplitude for Mössbauer neutrinos

Assume atoms bound in harmonic oscillator potentials,
e.g. for ^{3}H in the source:

$$\psi_{H,S}(\vec{x}, t) = \left[\frac{m_H \omega_{H,S}}{\pi} \right]^{\frac{3}{4}} \exp\left[-\frac{1}{2} m_H \omega_{H,S} |\vec{x} - \vec{x}_S|^2 \right] \cdot e^{-iE_{H,s}t}$$

Transition amplitude for Mössbauer neutrinos (contd.)

$$i\mathcal{A} = \int d^3x_1\, dt_1 \int d^3x_2\, dt_2 \left(\frac{m_H \omega_{H,S}}{\pi}\right)^{\frac{3}{4}} \exp\left[-\frac{1}{2} m_H \omega_{H,S} |\vec{x}_1 - \vec{x}_S|^2\right] e^{-iE_{H,S} t_1}$$

$$\cdot \left(\frac{m_{He} \omega_{He,S}}{\pi}\right)^{\frac{3}{4}} \exp\left[-\frac{1}{2} m_{He} \omega_{He,S} |\vec{x}_1 - \vec{x}_S|^2\right] e^{+iE_{He,S} t_1}$$

$$\cdot \left(\frac{m_{He} \omega_{He,D}}{\pi}\right)^{\frac{3}{4}} \exp\left[-\frac{1}{2} m_{He} \omega_{He,D} |\vec{x}_2 - \vec{x}_D|^2\right] e^{-iE_{He,D} t_2}$$

$$\cdot \left(\frac{m_H \omega_{H,D}}{\pi}\right)^{\frac{3}{4}} \exp\left[-\frac{1}{2} m_H \omega_{H,D} |\vec{x}_2 - \vec{x}_D|^2\right] e^{+iE_{H,D} t_2}$$

$$\cdot \sum_j \mathcal{M}^\mu \mathcal{M}^{\nu *} |U_{ej}|^2 \int \frac{d^4 p}{(2\pi)^4} e^{-ip_0(t_2 - t_1) + i\vec{p}(\vec{x}_2 - \vec{x}_1)}$$

$$\cdot \bar{u}_{e,S} \gamma_\mu (1 - \gamma^5) \frac{i(\not{p} + m_j)}{p_0^2 - \vec{p}^2 - m_j^2 + i\epsilon} (1 + \gamma^5) \gamma_\nu u_{e,D}.$$

- $dt_1\, dt_2$-integrals $\rightarrow$ energy-conserving δ functions $\rightarrow$ p_0-integral trivial
- $d^3x_1\, d^3x_2$-integrals are Gaussian
- d^3p-integral: Use Grimus-Stockinger theorem (limit of propagator for large $L = |\vec{x}_D - \vec{x}_S|$). W. Grimus, P. Stockinger, Phys. Rev. **D54** (1996) 3414, hep-ph/9603430

From the amplitude to the transition rate

Amplitude:

$$i\mathcal{A} = \frac{-i}{2L}\mathcal{N}\,\delta(E_S - E_D)\,\exp\left[-\frac{E_S^2 - m_j^2}{2\sigma_p^2}\right]\sum_j \mathcal{M}^\mu\mathcal{M}^{\nu*}|U_{ej}|^2\,e^{i\sqrt{E_S^2 - m_j^2}\,L}$$

$$\cdot\,\bar{u}_{e,S}\gamma_\mu\frac{1-\gamma^5}{2}(\slashed{p}_j + m_j)\frac{1+\gamma^5}{2}\gamma_\nu u_{e,D},$$

$$\sigma_p^{-2} = (m_{\mathrm{H}}\omega_{\mathrm{H},S} + m_{\mathrm{He}}\omega_{\mathrm{He},S})^{-1} + (m_{\mathrm{H}}\omega_{\mathrm{H},D} + m_{\mathrm{He}}\omega_{\mathrm{He},D})^{-1}$$

From the amplitude to the transition rate

Amplitude:

$$i\mathcal{A} = \frac{-i}{2L}\mathcal{N}\,\delta(E_S - E_D)\,\exp\left[-\frac{E_S^2 - m_j^2}{2\sigma_p^2}\right]\sum_j \mathcal{M}^\mu \mathcal{M}^{\nu*}|U_{ej}|^2\,e^{i\sqrt{E_S^2 - m_j^2}L}$$

$$\cdot\,\bar{u}_{e,S}\gamma_\mu \frac{1-\gamma^5}{2}(\not{p}_j + m_j)\frac{1+\gamma^5}{2}\gamma_\nu u_{e,D},$$

$$\sigma_p^{-2} = (m_H\omega_{H,S} + m_{He}\omega_{He,S})^{-1} + (m_H\omega_{H,D} + m_{He}\omega_{He,D})^{-1}$$

Transition rate: Integrate $|\mathcal{A}|^2$ over densities of initial and final states

$$\Gamma \propto \int_0^\infty dE_{H,S}\,dE_{He,S}\,dE_{He,D}\,dE_{H,D}$$

$$\cdot\,\delta(E_S - E_D)\rho_{H,S}(E_{H,S})\,\rho_{He,D}(E_{He,D})\,\rho_{He,S}(E_{He,S})\,\rho_{H,D}(E_{H,D})$$

$$\cdot\,\sum_{j,k}|U_{ej}|^2|U_{ek}|^2\,\underbrace{\exp\left[-\frac{2E_S^2 - m_j^2 - m_k^2}{2\sigma_p^2}\right]}_{\substack{\text{Analogue of Lamb-Mössbauer factor} \\ \text{(Probability of recoil-free transition)}}}\,\underbrace{e^{i\left(\sqrt{E_S^2 - m_j^2} - \sqrt{E_S^2 - m_k^2}\right)L}}_{\text{Oscillation phase}}$$

Inhomogeneous line broadening

Energy levels of ^{3}H and ^{3}He in the source and detector are smeared e.g. due to crystal impurities, lattice defects, etc.

R. S. Raghavan, hep-ph/0601079, W. Potzel, Phys. Scripta **T127** (2006) 85, B. Balko, I. W. Kay, J. Nicoll, J. D. Silk, G. Herling, Hyperfine Int. **107** (1997) 283

Good approximation for distribution of energy levels:

$$\rho_{A,B}(E_{A,B}) = \frac{\gamma_{A,B}/2\pi}{(E_{A,B} - E_{A,B,0})^2 + \gamma_{A,B}^2/4}$$

Mössbauer neutrino transition rate for two neutrino flavours ($m_2 > m_1$):

$$\Gamma \propto \exp\left[-\frac{E_{S,0}^2 - m_2^2}{\sigma_p^2}\right] \exp\left[-\frac{|\Delta m^2|}{2\sigma_p^2}\right] \frac{(\gamma_S + \gamma_D)/2\pi}{(E_{S,0} - E_{D,0})^2 + \frac{(\gamma_S + \gamma_D)^2}{4}}$$
$$\cdot \left\{1 - 2s^2c^2\left[1 - \frac{1}{2}(e^{-L/L_S^{\mathrm{coh}}} + e^{-L/L_D^{\mathrm{coh}}})\cos\left(\pi\frac{L}{L^{\mathrm{osc}}}\right)\right]\right\}$$

- Oscillation term
- Lamb-Mössbauer factor
- Breit-Wigner resonance term
- Coherence terms with $L_{S,D}^{\mathrm{coh}} = 4\bar{E}^2/\Delta m^2\gamma_{S,D}$ ($\gg L^{\mathrm{osc}}$ in reality)
- Note: No localization term $\sim \exp[-2\pi^2\xi^2(1/2\sigma_p L_{jk}^{\mathrm{osc}})^2]$ since $E_i \simeq E_j$.

Homogeneous line broadening

- Fluctuating electromagnetic fields in solid state crystal
  - Fluctuating energy levels of ^{3}H and ^{3}He.
- Classical Mössbauer effect: Homogeneous and inhomogeneous broadening both lead to Lorentzian line shapes
  - Experimentally indistinguishable
  - We expect a result similar to that for the case of inhomogeneous broadening (Rember Kiers, Nussinov, Weiss!)
- Ansatz: Introduce modulation factors of the form

$$f_{A,B}(t) = \exp\left[-i \int_0^t dt' \left[E_{A,B}(t') - E_{A,B,0} \right] t' \right]$$

in the ^{3}H and ^{3}He wave functions ($A = \text{H}, \text{He}$, $B = S, D$).

J. Odeurs, Phys. Rev. **B52** (1995) 6166

Transition amplitude for homogeneous line broadening

$$
i\mathcal{A} = \int d^3x_1 \, dt_1 \int d^3x_2 \, dt_2 \left(\frac{m_{\mathrm{H}}\omega_{\mathrm{H},S}}{\pi}\right)^{\frac{3}{4}} \exp\left[-\frac{1}{2}m_{\mathrm{H}}\omega_{\mathrm{H},S}|\vec{x}_1 - \vec{x}_S|^2\right] e^{-iE_{\mathrm{H},S}t_1}
$$

$$
\cdot \left(\frac{m_{\mathrm{He}}\omega_{\mathrm{He},S}}{\pi}\right)^{\frac{3}{4}} \exp\left[-\frac{1}{2}m_{\mathrm{He}}\omega_{\mathrm{He},S}|\vec{x}_1 - \vec{x}_S|^2\right] e^{+iE_{\mathrm{He},S}t_1}
$$

$$
\cdot \left(\frac{m_{\mathrm{He}}\omega_{\mathrm{He},D}}{\pi}\right)^{\frac{3}{4}} \exp\left[-\frac{1}{2}m_{\mathrm{He}}\omega_{\mathrm{He},D}|\vec{x}_2 - \vec{x}_D|^2\right] e^{-iE_{\mathrm{He},D}t_2}
$$

$$
\cdot \left(\frac{m_{\mathrm{H}}\omega_{\mathrm{H},D}}{\pi}\right)^{\frac{3}{4}} \exp\left[-\frac{1}{2}m_{\mathrm{H}}\omega_{\mathrm{H},D}|\vec{x}_2 - \vec{x}_D|^2\right] e^{+iE_{\mathrm{H},D}t_2}
$$

$$
\cdot \sum_j \mathcal{M}_S^\mu \mathcal{M}_D^{\nu*} |U_{ej}|^2 \int \frac{d^4p}{(2\pi)^4} \exp\left[-ip_0(t_2 - t_1) + i\vec{p}(\vec{x}_2 - \vec{x}_1)\right]
$$

$$
\cdot \bar{u}_{e,S}\gamma_\mu(1 - \gamma^5) \frac{i(\slashed{p} + m_j)}{p_0^2 - \vec{p}^2 - m_j^2 + i\epsilon} (1 + \gamma^5)\gamma_\nu u_{e,D}
$$

Transition amplitude for homogeneous line broadening

$$i\mathcal{A} = \int d^3x_1 \, dt_1 \int d^3x_2 \, dt_2 \left(\frac{m_H \omega_{H,S}}{\pi}\right)^{\frac{3}{4}} \exp\left[-\frac{1}{2} m_H \omega_{H,S} |\vec{x}_1 - \vec{x}_S|^2\right] f_{H,S}(t_1) \, e^{-iE_{H,S} t_1}$$

$$\cdot \left(\frac{m_{He} \omega_{He,S}}{\pi}\right)^{\frac{3}{4}} \exp\left[-\frac{1}{2} m_{He} \omega_{He,S} |\vec{x}_1 - \vec{x}_S|^2\right] f_{He,S}^*(t_1) \, e^{+iE_{He,S} t_1}$$

$$\cdot \left(\frac{m_{He} \omega_{He,D}}{\pi}\right)^{\frac{3}{4}} \exp\left[-\frac{1}{2} m_{He} \omega_{He,D} |\vec{x}_2 - \vec{x}_D|^2\right] f_{He,D}(t_2) \, e^{-iE_{He,D} t_2}$$

$$\cdot \left(\frac{m_H \omega_{H,D}}{\pi}\right)^{\frac{3}{4}} \exp\left[-\frac{1}{2} m_H \omega_{H,D} |\vec{x}_2 - \vec{x}_D|^2\right] f_{H,D}^*(t_2) \, e^{+iE_{H,D} t_2}$$

$$\cdot \sum_j \mathcal{M}_S^\mu \mathcal{M}_D^{\nu*} |U_{ej}|^2 \int \frac{d^4p}{(2\pi)^4} \exp\left[-ip_0(t_2 - t_1) + i\vec{p}(\vec{x}_2 - \vec{x}_1)\right]$$

$$\cdot \bar{u}_{e,S} \gamma_\mu (1 - \gamma^5) \frac{i(\slashed{p} + m_j)}{p_0^2 - \vec{p}^2 - m_j^2 + i\epsilon} (1 + \gamma^5) \gamma_\nu u_{e,D}$$

Transition amplitude for homogeneous line broadening

$$i\mathcal{A} = \int d^3x_1 \, dt_1 \int d^3x_2 \, dt_2 \left(\frac{m_H \omega_{H,S}}{\pi} \right)^{\frac{3}{4}} \exp\left[-\frac{1}{2} m_H \omega_{H,S} |\vec{x}_1 - \vec{x}_S|^2 \right] f_{H,S}(t_1) \, e^{-iE_{H,S} t_1}$$

$$\cdot \left(\frac{m_{He} \omega_{He,S}}{\pi} \right)^{\frac{3}{4}} \exp\left[-\frac{1}{2} m_{He} \omega_{He,S} |\vec{x}_1 - \vec{x}_S|^2 \right] f^*_{He,S}(t_1) \, e^{+iE_{He,S} t_1}$$

$$\cdot \left(\frac{m_{He} \omega_{He,D}}{\pi} \right)^{\frac{3}{4}} \exp\left[-\frac{1}{2} m_{He} \omega_{He,D} |\vec{x}_2 - \vec{x}_D|^2 \right] f_{He,D}(t_2) \, e^{-iE_{He,D} t_2}$$

$$\cdot \left(\frac{m_H \omega_{H,D}}{\pi} \right)^{\frac{3}{4}} \exp\left[-\frac{1}{2} m_H \omega_{H,D} |\vec{x}_2 - \vec{x}_D|^2 \right] f^*_{H,D}(t_2) \, e^{+iE_{H,D} t_2}$$

$$\cdot \sum_j \mathcal{M}_S^\mu \mathcal{M}_D^{\nu*} |U_{ej}|^2 \int \frac{d^4p}{(2\pi)^4} \exp\left[-ip_0(t_2 - t_1) + i\vec{p}(\vec{x}_2 - \vec{x}_1) \right]$$

$$\cdot \bar{u}_{e,S} \gamma_\mu (1 - \gamma^5) \frac{i(\slashed{p} + m_j)}{p_0^2 - \vec{p}^2 - m_j^2 + i\epsilon} (1 + \gamma^5) \gamma_\nu u_{e,D}$$

- $d^3x_1 \, d^3x_2$-integrals are Gaussian
- d^3p-integral: Use Grimus-Stockinger theorem.
- Transition rate $\Gamma \propto \langle \mathcal{A}\mathcal{A}^* \rangle$ (average of $\mathcal{A}\mathcal{A}^*$ over all possible ^{3}H and ^{3}He states) can be computed if fluctuations are Markovian (system has no memory)

Transition rate for homogeneous line broadening

Result:

$$\Gamma \propto \exp\left[-\frac{E_{S,0}^2 - m_2^2}{\sigma_p^2}\right] \exp\left[-\frac{|\Delta m^2|}{2\sigma_p^2}\right] \frac{(\gamma_S + \gamma_D)/2\pi}{(E_{S,0} - E_{D,0})^2 + \frac{(\gamma_S + \gamma_D)^2}{4}}$$

$$\cdot \left\{1 - 2s^2c^2\left[1 - \frac{1}{2}(e^{-L/L_S^{\mathrm{coh}}} + e^{-L/L_D^{\mathrm{coh}}})\cos\left(\pi\frac{L}{L^{\mathrm{osc}}}\right)\right]\right\}$$

. . . identical to the result for inhomogeneous line broadening.

> This illustrates that ...
>
> A continuous flux of identical wave packets cannot be distinguished from an ensemble of plane waves with the same momentum distribution.
> (Neutrino density matrices for the two ensembles are identical)
>
> K. Kiers, S. Nussinov, N. Weiss, Phys. Rev. **D53** (1996) 537, hep-ph/9506271

Plane waves vs. wave packets

We know: Plane wave calculations yield the correct oscillation formula . . .
. . . even though they are plagued by inconsistencies

- Plane waves are infinitely delocalized, but concept of "baseline" requires localization
- Relation $L \sim T$ (required in the derivation of the oscillation probability unless $E_i \sim E_j$) cannot be justified.
- Perfect knowledge of E_j and p_j means we know m_j
  $\rightarrow$ no oscillations B. Kayser Phys. Rev. **D24** (1981) 110
- . . .

So, why do plane wave approaches work so well?!?

Reason 1: Statistical ensembles of neutrinos

A continuous flux of identical wave packets cannot be distinguished from an ensemble of plane waves with the same momentum distribution.
(Neutrino density matrices for the two ensembles are identical)

K. Kiers, S. Nussinov, N. Weiss, Phys. Rev. **D53** (1996) 537, hep-ph/9506271

For a time-independent flux of neutrinos:

- Wave packet picture equivalent to plane wave picture
- Production and detection can still be localized, i.e. density matrix $\rho(x)$ is probed at fixed $x = L$.
- Density matrix is diagonal in
  $\rightarrow$ only states with same energy ($E_i = E_j$) can interfere
  $\rightarrow$ no T-dependence, relation $L \sim T$ not required.
- No perfect knowledge of E_j and p_j for each individual neutrino
  $\rightarrow$ oscillations still possible

Illustration

If time-of-flight information is not available, only energy eigenstates interfere.

- Consider neutrino density matrix

$$\rho \equiv \sum_{\text{many neutrinos}} |\psi_n\rangle\langle\psi_n|$$

- von Neumann equation:

$$\dot{\rho} = -i[H, \rho]$$

- Stationary (time-independent) neutrino flux: $\dot{\rho} = 0$
  $\rightarrow \rho$ and H commute; ρ can be written as ensemble of energy eigenstates

$$\rho = \int dE \, c(E) \, |E\rangle\langle E| \qquad (\star)$$

- If $\dot{\rho} \neq 0$, but time information is discarded ($\int dt \, \rho$), energy-off-diagonal elements (containing factors $e^{i(E_i - E_j)t}$) will be averaged to zero.
  $\rightarrow$ same situation as for stationary flux, $(\star)$ holds.

Reason 2: Factorizing out wave packet effects

Consider Gaussian wave packet:

$$\psi_\alpha(\vec{x}, t) \propto \sum_j U_{\alpha j}^* e^{-iE_{j0}t + i\vec{p}_{j0}\vec{x}} \exp\left[\frac{(\vec{x} - \vec{v}_j t)^2}{4\sigma_x^2}\right]$$

$\rightarrow$ Plane wave, multiplied with shape factor

At a given spacetime point, phase difference depends only on average energies E_{j0} and momenta p_{j0}.
$\rightarrow$ Oscillation length cannot depend on wave packet shape and width.

(Only wave packet components located at the same spacetime point can interfere.)

The wave packet approach shows that the assumptions made in the plane wave picture are justified and consistent.

Outline

Entanglement in neutrino oscillations

Consider two-body decay $\pi \to \mu + \nu_\mu$.

Question: Are ν_μ and μ kinematically entangled?
Should the final state wave function be written as

$$\sum_j U^*_{\mu j} |\mu_j(P_j, X)\rangle \, |\nu_j(p_j, x)\rangle \ ?$$

YES!

- Without observation, all possible final states are produced simultaneously.
- Entanglement used successfully in *b* physics

NO!

- Energy-momentum uncertainty implies that every *p* can come with many different *P* (wave packets!).
- Only Feynman diagrams with identical external legs can interfere.

Cohen Glashow Ligeti; Akhmedov Smirnov; Kayser Kopp Robertson Vogel

The entangled point of view

Describe particles as plane waves with precisely known 4-momenta:

$$\pi(p_\pi) \to \mu(P_j) + \nu_j(p_j)$$

where $p_i \neq p_j$, $P_i \neq P_j$ due to E–p conservation.

- Energy and momentum uncertainties in the production/detection processes imply that there is a spectrum of p_π, p_i, P_i.
- Uncertainties allow interference between states with different p_i, P_i.
- The contribution of $\mu(P_j)$ to the phase of the entangled state is the same for all P_j: Akhmedov Smirnov 1008.2077

$$\begin{aligned}
&\big[E(P_i) - E(P_j)\big]\,T - \big[P_i - P_j\big]\,L \\
&= \big[E_0 + \delta E_i - E_0 - \delta E_j\big]\,T - \Big[E_0 + \delta E_i \frac{E_0}{P_0} - E_0 - \delta E_j \frac{E_0}{P_0}\Big]\,L \\
&= \big[\delta E_i - \delta E_j\big]\,(T - L/v) \simeq 0
\end{aligned}$$

$\to$ the charged lepton does not oscillate.

The wavepacket point of view

$P_f(k)$ $D_f(k')$

ν

$P_i(q)$ $D_i(q')$
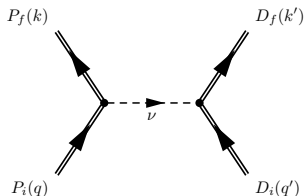
- External particles (parent pion, nucleus in the detector, . . .) are observed
  $\rightarrow$ their state at $t \rightarrow \pm\infty$ is fixed by the observation

- Each neutrino mass eigenstate may interact with different momentum components of the external particles' wave packets.
  (in a sense, there is entanglement between the neutrino and *components* of the wave packet. This is automatically taken care of by the *E–p* conserving δ-functions at the Feynman vertices.)

Outline

Neutrino flavor eigenstates

Flavor fields defined as $\nu_\alpha = U_{\alpha j}\nu_j$.

The problem: Flavor f_α does not commute with the Hamiltonian. (for $E_i \neq E_j$)

$$
\begin{aligned}
\left[\mathcal{F}, H\right] &\sim \left[\sum_{\alpha=e,\mu,\tau} f_\alpha |\nu_\alpha\rangle\langle\nu_\alpha|, \sum_{k=1,2,3} E_k |\nu_k\rangle\langle\nu_k|\right] \\
&= \sum_{\alpha=e,\mu,\tau} f_\alpha \left[\sum_{i,j=1,2,3} U_{\alpha i}^* U_{\alpha j} |\nu_i\rangle\langle\nu_j|, \sum_{k=1,2,3} E_k |\nu_k\rangle\langle\nu_k|\right] \\
&= \sum_{\alpha=e,\mu,\tau} f_\alpha \sum_{i,j=1,2,3} U_{\alpha i}^* U_{\alpha j}\left(E_j |\nu_i\rangle\langle\nu_j| - E_i |\nu_j\rangle\langle\nu_i|\right)
\end{aligned}
$$

Therefore, flavor eigenstates are not meaningful tools for calculating transition amplitudes.

Outline

1 Motivation

2 Wave packet approaches to neutrino oscillations
- QM: Neutrino wave packets
- QFT: Feynman diagrams with external wave packets
- The connection between wave packet and plane wave approaches

3 Entanglement in neutrino oscillations

4 Neutrino flavor eigenstates

5 Summary and conclusions

Answers

- What happens to energy-momentum conservation in neutrino oscillations?
  - In the wave packet approach, average momenta need not be conserved. For each momentum component of the wave packets, E–p conservation holds.
  - No energy-momentum non-conservation beyond what is allowed by the Heisenberg uncertainties.
  - If coherent superposition of neutrino mass eigenstates is forbidden by E–p conservation (hypothetical experiment with excellent E and p resolutions), oscillations vanish (ensured by localization terms).

- Under what conditions is coherence between different neutrino mass eigenstates lost?
  - For $L \gg L^{\mathrm{coh}}$, wave packets separate $\rightarrow$ loss of coherence (equivalently: Poor energy resolution smears out oscillations)
  - No coherence if experiment allows determination of neutrino mass

- Is the standard oscillation formula universal, or are there situations where it needs to be modified?
  - No situation with observable deviations has been found.
  - Mössbauer neutrinos oscillate in the standard way.
  - Entanglement does not lead to new phenomena.

Answers (contd.)

- Why does the standard oscillation formula work so well, even though its derivation is plagued by inconsistencies?
  - ▶ Wave packet effects can be factorized out.
  - ▶ Statistical ensemble of wave packets equivalent to statistical ensemble of plane waves. No inconsistencies in this case.

- Are the neutrino and its interaction partners in an entangled state? If so, what are the phenomenological consequences of entanglement?
  - ▶ Entanglement is a valid, but unnecessary concept.
  - ▶ No phenomenological consequences of entanglement.

- Are flavor eigenstates well-defined and useful objects in QFT?
  - ▶ Flavor eigenstates can be defined, but are useless for matrix element calculations because they do not commute with H.

Thank you!