Recurrence and Ergodicity of Interacting Particle Systems

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Abstract

Many interacting particle systems with short range interactions are not ergodic, but converge weakly towards a mixture of their ergodic invariant measures. The question arises whether a.s. the process eventually stays close to one of these ergodic states, or if it changes between the attainable ergodic states infinitely often ("recurrence"). Under the assumption that there exists a convergence–determining class of distributions that is (strongly) preserved under the dynamics, we show that the system is in fact recurrent in the above sense.

We apply our method to several interacting particle systems, obtaining new or improved recurrence results. In addition, we answer a question raised by Ed Perkins concerning the change of the locally predominant type in a model of mutually catalytic branching.

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1 Introduction

We start with an example to explain the problem we address in this work.

Consider the (basic) voter model $(\xi_t)_{t\geq 0}$ on $\mathbb{Z}^d$. This is a Markov process $(\xi_t)_{t\geq 0}$ on $X = \{0,1\}^{\mathbb{Z}^d}$, equipped with the product topology. Think of each point in $\mathbb{Z}^d$ as being occupied by an individual that is capable of holding either of the opinions 0 and 1. After a rate-one exponential waiting time, a given individual chooses one of his $2d$ nearest neighbors at random and assumes its opinion. All waiting times and choices of neighbors are made independently. The opinion of the voter at x at time t is given by $\xi_t(x)$.

For more a detailed description and background information, see Chapter V of Liggett (1985).

It is well known that the voter model clusters in dimension $d \leq 2$. More precisely, if we start at time 0 with independent opinions, where opinion 1 has probability $\theta \in (0,1)$, then

$$\mathcal{L}[\xi_t] \overset{t\to\infty}{\Longrightarrow} (1-\theta)\delta_{\mathbf{0}} + \theta\delta_{\mathbf{1}}. \tag{1.1}$$

Here, $\delta_{\mathbf{0}}$ and $\delta_{\mathbf{1}}$ are the unit masses on the states where all individuals have opinion 0, respectively 1, $\mathcal{L}$ denotes the law of a random variable, and $\Longrightarrow$ denotes weak convergence of probability measures. Note that since X carries the product topology, (1.1) is equivalent to convergence of the finite dimensional distributions. A question that arises naturally, given (1.1), is:

Does the opinion at a given site change value infinitely often?

The question has been answered affirmatively by means of rather special arguments in Cox and Griffeath (1986).

A simple argument that works for shift ergodic initial states was brought to our attention by Jeff Steif: Consider the events

$$A_i = \{\exists\, T:\ \xi_t(i) = 1,\ t \geq T\}, \qquad i \in \mathbb{Z}^d.$$

For $|i-j| = 1$ is is easy to see that $A_i = A_j$ a.s. hence a.s. $A_i = A := \cap_j A_j$. However A is shift invariant and by ergodicity we have $\mathbf{P}[A] \in \{0,1\}$. Since $\theta < 1$, clearly $\mathbf{P}[A] = 0$. Now change the 1 in the definition of A_i into 0 to conclude that the opinion changes infinitely often.

There are two drawbacks of this argument: (i) It works only for shift ergodic initial states. (ii) For many models it is hard to check whether $A_i = A_j$ a.s. or not.

The aim of this work is to give a robust and simple abstract argument that can be applied to a large variety of models and for initial states that only need to have a global density. We do not assume translation invariance or even ergodicity. In particular, our argument does not rely on quantitative estimates that make use of special features (or the dimension!) of the considered models. The basic idea is simple: (i) Find a certain class of probability measures on the state space that is preserved under the dynamics of (ξ_t), and that ensures convergence to the limiting state in an appropriate sense. (ii) Starting in this class, there is a positive probability that at a large fixed time the chain is close to a given extremal state ($\delta_{\mathbf{0}}$ or $\delta_{\mathbf{1}}$ in the case of the voter model). If the chain is not close to that state, then since the chain is still in the given class, one tries again using the Markov property. Eventually, success will occur, and hence will occur infinitely often.

The difficulty lies in finding the right notion of convergence for step (i).

For the voter model, we are able to prove a.s. alternation of types under more general conditions than were considered in Cox and Griffeath (1986). We also consider several related models, as well as a model of mutually catalytic branching recently introduced by Dawson and Perkins (1998).

Note that our focus lies on the situation where the model is not ergodic, i.e., the weak limit points are *mixtures* of the ergodic invariant measures. We show that the process gets “close” to any of the ergodic states (occurring in the mixture) at arbitrarily late times. This question has often been connected to the question of whether low-dimensional binary branching random walk, starting in a homogeneous Poisson field, populates a given site at arbitrarily late times. This is known to be be true for $d = 2$ and false for $d = 1$. Note, however, that this is a fundamentally different question than the one we address, since branching random walk is ergodic. Namely, if $d \leq 2$, the unit mass $\delta_{\mathbf{0}}$ on the empty configuration is the only invariant measure (with σ-finite intensity measure).

As mentioned above the correct notion of convergence of random probability measures is crucial. We discuss the topological details and give the abstract statement in Section 2. In Section 3 we apply our result to:

- the multitype voter model,
- interacting diffusions on $[0, 1]$,
- interacting Fleming Viot Processes,
- interacting Brownian motions,
- mutually catalytic branching super random walk.

We would like to close this introduction by mentioning that there are non–trivial examples where instead of recurrence of both (all) types there is almost sure *fixation*.

In the so–called n–type cyclic voter model, each individual is capable of holding one of the opinions $0, \dots, n-1$. As above, after an exponential waiting time, the voter at i chooses a nearest-neighbor j, but now adopts the opinion at j only if $x(j) - x(i) = 1 \pmod n$. Bramson and Griffeath (1989) show that for $d = 1$, starting from a symmetric product measure, this model fixates if and only if $n \geq 5$.

For a voter model in a certain random environment, Fontes, Isopi and Newman (1999) show that recurrence occurs if $d = 1$, and that there is a.s. fixation if $d \geq 2$. A related model is the so–called disordered Ising model at zero temperature (see Nanda, Newman and Stein (1999) and also Newman and Stein (1999)). Started from a deep quench (this is symmetric product measure in the physicists' language), the occurrence of fixation or recurrence depends on the details of the disorder (random environment). For the homogeneous model (no disorder), there is recurrence if $d \geq 2$. The problem is still open for $d \geq 3$.

2 Result

In this section we formulate and prove our abstract result.

Let X be a locally compact Polish space and denote by $\mathcal{P}(X)$ the space of probability measures on X equipped with the topology of weak convergence of probability measures. $\mathcal{P}(X)$ is again a locally compact Polish space (see, e.g., Kallenberg (1983)). Consider now a discrete time Markov process $(\xi_n)_{n \in \mathbb{N}_0}$ on X. (We could consider a Feller process $(\xi_t)_{t \geq 0}$ on X instead, but we choose the discrete time setting for the sake of generality.) Denote by $(\mathcal{S}(n))_{n \in \mathbb{N}_0}$ its semigroup. That is, for $\mu \in \mathcal{P}(X)$ and $n \in \mathbb{N}_0$,

$$\mu \mathcal{S}(n) = \mathcal{L}^\mu[\xi_n].$$

We want to describe the longtime behavior of (ξ_n) in terms of its possible limit points μ_θ, $\theta \in \Theta$, where Θ is an abstract set. (We do not assume that $(\mu_\theta)_{\theta \in \Theta}$ necessarily exhausts the class of possible limit points.) In the example of the voter model, $\Theta = [0, 1]$ and $\mu_\theta = (1-\theta)\delta_{\mathbf{0}} + \theta \delta_{\mathbf{1}}$.

Now we make the crucial definition:

Definition 2.1 *The domain of stochastic attraction $\mathcal{D}(\mu_\theta)$ of μ_θ is the set of measures $\mu \in \mathcal{P}(X)$ such that for all $m \in \mathbb{N}_0$, $\mathbf{P}^\mu[\xi_m \in dx]$-stochastically, $\mathcal{L}^x[\xi_n] \overset{n\to\infty}{\Longrightarrow} \mu_\theta$. Formally,*

$$\mathcal{D}(\mu_\theta) = \Big\{ \mu \in \mathcal{P}(X) : \ \mathbf{P}^\mu[\mathcal{L}^{\xi_m}[\xi_n] \in \mathfrak{U}] \overset{n\to\infty}{\longrightarrow} 1 \ \forall \textit{ open } \mathfrak{U} \ni \mu_\theta \textit{ and } m \in \mathbb{N}_0 \Big\}. \tag{2.1}$$

Clearly, $\mathcal{D}(\mu_\theta)$ is a convex set but it is in general not closed. For example, in the voter model $\pi_\theta \mathcal{S}(n) \overset{n\to\infty}{\Longrightarrow} \mu_\theta$, where π_θ is the product measure on $\{0,1\}^{\mathbb{Z}^d}$ with intensity θ. We will see later that $\pi_\theta \mathcal{S}(n) \in \mathcal{D}(\mu_\theta)$, $n \in \mathbb{N}_0$, but obviously $\mu_\theta \notin \mathcal{D}(\mu_\theta)$ if $\theta \in (0,1)$. Since $\mathcal{D}(\mu_\theta)$ is not compact we cannot hope for a nice description in terms of extremal elements. In spite of this, we give a mild sufficient condition for a set $\mathcal{M}_\theta \subset \mathcal{P}(X)$ to be a subset of $\mathcal{D}(\mu_\theta)$ that covers a wide range in the examples.

Assumption 1. $\mathcal{M}_\theta \subset \mathcal{P}(X)$ is invariant under the dynamics of (ξ_n), i.e.,

$$\mathcal{M}_\theta \mathcal{S}(1) \subset \mathcal{M}_\theta. \tag{A1}$$

Assumption 2. For all $\mu \in \mathcal{M}_\theta$, the laws $\mathcal{L}^x[\xi_n]$ converge to μ_θ in $\mu(dx)$ probability. That is, if $\mathfrak{U} \subset \mathcal{P}(X)$ is open and $\mu_\theta \in \mathfrak{U}$, then for all $\mu \in \mathcal{M}_\theta$,

$$\mu(\{x \in X :\ \mathcal{L}^x[\xi_n] \in \mathfrak{U}\}) \overset{n\to\infty}{\longrightarrow} 1. \tag{A2}$$

Proposition 2.2 *If $\mathcal{M}_\theta$ fulfills the assumption (A1) and (A2) then $\mathcal{M}_\theta \subset \mathcal{D}(\mu_\theta)$. Further, $\mathcal{D}(\mu_\theta)$ fulfills (A1) and (A2) and is hence the maximal set fulfilling (A1) and (A2).*

Proof The proof is simple and is left as an exercise. □

In the context of the voter model, for $0 < \theta < 1$ and with $\mu_\theta = (1-\theta)\mu_{\mathbf{0}} + \theta\mu_{\mathbf{1}}$, we would like to argue that (A1) and (A2) guarantee that for any initial measure $\mu \in \mathcal{M}_\theta$, the process ξ_t gets "close" to **0** and to **1** at arbitrarily late times t. The meaning of (A1) is clear. However, condition (A2) is somewhat unusual, so we would like to discuss its interpretation and our reasons for choosing it.

There are basically three types of convergence that we might choose for (A2): convergence of the means, stochastic convergence and almost sure convergence. Since we consider convergence of random probability measures, convergence in L^1 and stochastic convergence coincide, and both are implied by almost sure convergence, while both imply convergence of the means. We illustrate the meaning of these concepts in the example of the voter model.

By *convergence of the means*, we mean the condition

$$\mu\mathcal{S}(n) \overset{n\to\infty}{\Longrightarrow} \mu_\theta \qquad \forall \mu \in \mathcal{M}_\theta. \tag{2.2}$$

That is, $\mathcal{M}_\theta$ is a subset of the domain of attraction of μ_θ. In the example of the voter model, we could set $\mathcal{M}_\theta = \{\mu_\theta\}$, in which case (2.2) would certainly hold. However, in this case we would have $\xi_n \equiv \xi_0$ a.s., so there would be no change of types at all. Hence, this notion is too rough for our purposes.

By *almost sure convergence*, we mean the condition

$$\mu(\{x \in X :\ \mathcal{L}^x[\xi_n] \overset{n\to\infty}{\Longrightarrow} \mu_\theta\}) = 1 \qquad \forall \mu \in \mathcal{M}_\theta. \tag{2.3}$$

(Since (2.3) does not hold for the voter model with $\mathcal{M}_\theta = \{\mu_\theta\}$, our objection to (2.2) does not apply.) Certainly (2.3) implies (2.2), but it is correspondingly more difficult to verify in any given example. For the voter model, by using *duality* (see Chapter V of Liggett (1985)), it is possible to verify (2.3) for some classes $\mathcal{M}_\theta$. However, verification becomes rather difficult for more complicated models, so we do not adopt this notion of convergence.

By *stochastic convergence*, we mean exactly (A2), which is a weaker condition than (2.3), but still strong enough for our purposes. For the voter model, to verify (A2), we only have to show that for all finite $H \subset \mathbb{Z}^d$ and $\varepsilon > 0$,

$$\mu(\{x \in X :\ \mathbf{P}^x[\xi_n(i) = 1\,\forall i \in H] > \theta - \varepsilon,\ \mathbf{P}^x[\xi_n(i) = 0\,\forall i \in H] > 1 - \theta - \varepsilon\}) \overset{n\to\infty}{\longrightarrow} 1. \tag{2.4}$$

This fact, for μ belonging to a large class $\mathcal{M}_\theta$, is easily proved using duality (see the proof of Theorem 1) below.

Now we come back to the general situation. Let $S_\theta = \mathrm{supp}(\mu_\theta)$ be the closed support of μ_θ. For a sequence $(x_n)_{n\in\mathbb{N}}$ in X let $\mathcal{A}((x_n)_{n\in\mathbb{N}})$ denote the set of accumulation points of $(x_n)_{n\in\mathbb{N}}$ in X.

Proposition 2.3 *For each $\theta \in \Theta$, for all $\mu \in \mathcal{D}(\mu_\theta)$,*

$$\mathbf{P}^\mu[S_\theta \subset \mathcal{A}((\xi_n)_{n\in\mathbb{N}})] = 1. \tag{2.5}$$

Proof Let $U \subset X$ be open, with $U \cap S_\theta \neq \emptyset$. We will show that there exists a sequence $t_n \uparrow \infty$ such that

$$\mathbf{P}^\mu[\xi_{t_n} \in U \text{ infinitely often }] = 1. \tag{2.6}$$

Clearly, (2.6) implies (2.5).

By our choice of U,

$$\delta := \mu_\theta(U) > 0.$$

Choose $\mathfrak{U} \subset \mathcal{P}(X)$ open, $\mu_\theta \in \mathfrak{U}$, such that $\nu(U) > \delta/2$ for all $\nu \in \mathfrak{U}$. By (A1) and (A2) we can choose a sequence $t_n \uparrow \infty$ such that

$$(\mu\mathcal{S}(t_n))\big(\{x \in X :\ \mathcal{L}^x[\xi_{t_{n+1}-t_n}] \in \mathfrak{U}\}\big) > 1 - 2^{-n}.$$

Denote by A_n the event

$$A_n = \{\xi_{t_n} \in U\}$$

and let

$$B_n = \big\{x \in X :\ \mathcal{L}^x[\xi_{t_n}] \in \mathfrak{U}\big\}.$$

If we let $\mathcal{F}_t = \sigma(\xi_0, \xi_1, \ldots, \xi_t)$ be the filtration induced by (ξ_t) then by the Markov property for $n \in \mathbb{N}$

$$\begin{aligned} \mathbf{P}^\mu\big[\mathbf{P}^\mu[A_n|\mathcal{F}_{t_{n-1}}] < \delta/2\big] &= \mu\mathcal{S}(t_{n-1})\big(\{x \in X :\ \mathbf{P}^x[\xi_{t_n - t_{n-1}} \in U] < \delta/2\}\big) \\ &\leq \mu\mathcal{S}(t_{n-1})\big(\{x \in X :\ \mathcal{L}^x[\xi_{t_n - t_{n-1}}] \notin \mathfrak{U}\}\big) \\ &\leq 2^{-n}. \end{aligned} \tag{2.7}$$

Hence $\mathbf{P}^\mu$–almost sure $\mathbf{P}^\mu[A_n|\mathcal{F}_{t_{n-1}}] \geq \delta/2$ for infinitely many $n \in \mathbb{N}$. In particular,

$$\mathbf{P}^\mu\left[\sum_{n=1}^\infty \mathbf{P}^\mu[A_n|\mathcal{F}_{t_{n-1}}] = \infty\right] = 1. \tag{2.8}$$

Now according to the conditional Borel–Cantelli lemma (see, e.g., Durrett (1996), Corollary 4.3.2)

$$\limsup_{n\to\infty} A_n = \left\{\sum_{n=1}^\infty \mathbf{P}^\mu[A_n|\mathcal{F}_{t_{n-1}}] = \infty\right\} \qquad (\text{mod } \mathbf{P}^\mu). \tag{2.9}$$

Hence

$$\mathbf{P}^\mu\left[\limsup_{n\to\infty} A_n\right] = 1,$$

which implies (2.6). □

3 Applications

The situation we have in mind is that of a “general” interacting particle system where a global variable, typically the density of particles, is preserved under the dynamics. The models we consider here have a number of features in common. The state space is $X \subset V^G$, equipped with the product topology, where the countably infinite Abelian group G (we exclude expressis verbis the possibility of G finite!) plays the role of the site space. V is the space of values that a local coordinate can assume. In the context of genealogical models, we have $V \subset [0,\infty)^E$ or $V \subset \mathcal{M}_f(E)$ (the finite measures on E), where E is a “type” space. For

$v \in V$ we interpret $v(e)$ as the number of particles of type $e \in E$. When V is compact we can, in fact, take $X = V^G$, but for non-compact V, we need to impose growth conditions on the coordinates. In all cases, the interaction of the coordinates will be described in terms of an irreducible random walk kernel $a(\cdot,\cdot)$ on G. The continuous time transition kernel a_t is defined by

$$a_t = e^{-t} \sum_{n=0}^{\infty} \frac{t^n}{n!} a^{(n)},$$

where $a^{(n)}$ is the n-step transition probability of a.

For $v \in V$, we let $\mathbf{v}$ denote the element $\mathbf{v} \in X$ such that $\mathbf{v}(g) = v$ for all $g \in G$. $\mathcal{P}$ always denotes the space of probability measures on a locally compact Polish space equipped with the weak topology.

3.1 The Multitype voter model

Fix a positive integer $c > 1$, the number of types (opinions), let $E = \{1, \ldots, c\}$ be the space of types, and let $V = \{\mathbb{1}_{\{e\}},\ e \in E\}$. Let $X = V^G$, and define, for $x \in X$ and $g, g', h \in G$,

$$x_{g,g'}(h) = \begin{cases} x(h), & h \neq g, \\ x(g'), & h = g. \end{cases} \tag{3.1}$$

We define the voter model $(\xi_t)_{t \geq 0}$ to be the Markov process on X with generator $\mathcal{G}$, where for $F : X \to \mathbb{R}$ depending on only finitely many coordinates,

$$\mathcal{G}F(x) = \sum_{g,h \in G} a(g,h) \left(F(x_{g,h}) - F(x) \right). \tag{3.2}$$

Define the simplex

$$\Theta = \mathcal{P}(E) = \Big\{ \theta : E \to [0,1] \text{ with } \sum_{e \in E} \theta(e) = 1 \Big\}, \tag{3.3}$$

and for $\theta \in \Theta$ let $\mathcal{M}_\theta$ be the collection of $\mu \in \mathcal{P}(X)$ such that for all $g \in G$ and $e \in E$,

$$\lim_{s \to \infty} \int \mu(dx) ((a_s x(g) - \theta)(e))^2 = 0. \tag{3.4}$$

In the case that $G = \mathbb{Z}^d$, the collection $\mathcal{M}_\theta$ contains all translation invariant, shift ergodic $\mu \in \mathcal{P}(X)$ satisfying $\int \mu(dx) x(0) = \theta$ (see pp. 180–181 of Cox, Greven and Shiga (1995) for the case $c = 2$). For $\theta \in \Theta$ define

$$\mu_\theta = \sum_{e \in E} \theta(e)\, \delta_{\mathbf{e}}, \tag{3.5}$$

and note that $S_\theta = \{\mathbf{e} : e \in E \text{ and } \theta(e) > 0.\}$

We assume that the symmetrized kernel $\widehat{a}$ given by

$$\widehat{a}(g,h) = \frac{a(g,h) + a(h,g)}{2}$$

is recurrent. It is well known that the voter model clusters in this situation. In particular, Theorem V.1.9 of Liggett (1985) implies that for all $\mu \in \mathcal{M}_\theta$,

$$\mathcal{L}^\mu[\xi_t] \overset{t \to \infty}{\Longrightarrow} \mu_\theta. \tag{3.6}$$

Theorem 1 *Let $\theta \in \Theta$ and $\mu \in \mathcal{M}_\theta$. Then for all $e \in E$ with $\theta(e) > 0$, and for all finite $H \subset G$, all the components $\xi_t(h)$, $h \in H$, simultaneously assume the value $\mathbb{1}_{\{e\}}$ at arbitrarily late times with probability one.*

Proof It suffices to verify that (A1) and (A2) hold, in which case $\mathcal{M}_\theta \subset \mathcal{D}(\mu_\theta)$ is in the domain of stochastic attraction of μ_θ and our conclusion is justified by Proposition 2.3. To do this, we make use of *duality* (see Chapter V of Liggett (1985)), which we briefly describe. Let $(\eta_t^g,\ g \in G)_{t\geq 0}$ be a system of rate one continuous time coalescing random walks on G, with step distribution $a(g,h)$. For each $g \in G$, η_t^g is a random walk started at g. The random walks η_t^g run independently until two of them meet, at which time the walks (instantly) coalesce, and after that move together. A special case of the duality relation (see (V.1.7) of Liggett (1985)) connecting η_t and ξ_t is: for all $x \in X$, finite $H \subset G$ and $v \in V$,

$$\mathbf{P}^x[\xi_t(h) = v,\ h \in H] = \mathbf{P}[x(\eta_t^h) = v,\ h \in H]. \tag{3.7}$$

Fix $\mu \in \mathcal{M}_\theta$ and $t > 0$. To verify (A1), we must show that for fixed g and e,

$$\mathbf{E}^\mu\left[(a_s\xi_t(g) - \theta)(e)^2\right] \overset{s\to\infty}{\longrightarrow} 0. \tag{3.8}$$

For $H \subset G$, let τ_H to be the first time at which all the random walks started in H have coalesced,

$$\tau_H = \inf\{t > 0 :\ \eta_t^g = \eta_t^h\ \forall\, g,h \in H\}. \tag{3.9}$$

Note that

$$\mathbf{Cov}^x[\xi_t(g)(e), \xi_t(h)(e)] \leq \mathbf{P}[\tau_{\{g,h\}} \leq t].$$

Hence

$$\begin{aligned}\mathbf{E}^\mu[(a_s\xi_t(g)(e) - \theta(e))^2] &- \int \mu(dx)(a_{s+t}x(g)(e) - \theta(e))^2 \\ &= \int \mu(dx)\mathbf{Var}^x[a_s\xi_t(g)(e)] \\ &= \int \mu(dx) \sum_{h,k\in G} a_s(g,h)a_s(g,k)\mathbf{Cov}^x[\xi_t(h)(e), \xi_t(k)(e)] \\ &\leq \sum_{h,k\in G} a_s(g,h)a_s(g,k)F_t(k-h),\end{aligned} \tag{3.10}$$

where $F_t(k-h) = \mathbf{P}[\tau_{\{h,k\}} \leq t]$. By the assumption that $\mu \in \mathcal{M}_\theta$, the second term on the left side above tends to 0 as $s \to \infty$. The right side also tends to 0 as $s \to \infty$, since G is infinite and a is irreducible, and since $F_t(h) \to 0$ as $|h| \to \infty$. (That is, for any sequence (G_n) of finite subsets of G such that $G_n \uparrow G$ as $n \to \infty$, $\sup\{F_t(h) : h \in G \setminus G_n\} \to 0$ as $n \to \infty$.)

In order to show that (A2) holds, it suffices to show that for $\varepsilon > 0$ and finite $H \subset G$,

$$\lim_{t\to\infty} \mu(\{x :\ \mathbf{P}^x[\xi_t(h) = \mathbb{1}_{\{e\}} \text{ for all } h \in H] > \theta(e) - \varepsilon \text{ for all } e \in E\}) = 1. \tag{3.11}$$

The set E of types is finite, so it suffices to prove that for each fixed $e \in E$,

$$\lim_{t\to\infty} \mu(\{x :\ \mathbf{P}^x[\xi_t(h) = \mathbb{1}_{\{e\}} \text{ for all } h \in H] > \theta(e) - \varepsilon\}) = 1. \tag{3.12}$$

Choose an arbitrary $g \in H$. By (3.7),

$$\begin{aligned}\mathbf{P}^x[\xi_t(h) = \mathbb{1}_{\{e\}} \text{ for all } h \in H] &\geq \mathbf{P}[x(\eta_t^g) = \mathbb{1}_{\{e\}}, \tau_H \leq t] \\ &\geq \mathbf{P}[x(\eta_t^g) = \mathbb{1}_{\{e\}}] - \mathbf{P}[\tau_H > t] \\ &= a_t x(g)(e) - \mathbf{P}[\tau_H > t].\end{aligned}$$

Since we have assumed that $\widehat{a}$ is recurrent, $\mathbf{P}[\tau_H > t] \to 0$ as $t \to \infty$. Therefore,

$$\begin{aligned}\mu(\{x : \mathbf{P}^x[\xi_t(h) = \mathbb{1}_{\{e\}} &\text{ for all } h \in H] \leq \theta(e) - \varepsilon\}) \\ &\leq \mu(\{x : (a_t x(g) - \theta)(e) \leq -(\varepsilon - \mathbf{P}[\tau_H > t])\}) \\ &\leq \int \mu(dx)((a_t x(g) - \theta)(e))^2/(\varepsilon - \mathbf{P}[\tau_H > t])^2 \\ &\longrightarrow 0\end{aligned}$$

as $t \to \infty$, on account of (3.4). $\square$

3.2 Interacting diffusions

Here we consider a two–type genealogical model with migration and resampling. We suppose that at each site $g \in G$ there is a large colony of individuals, and each individual must be one of two genealogical types, A or B. The frequency of type A at site g at time t is $\xi_t(g)$. Hence $E = \{1,2\}$ and we identify $\mathcal{P}(E)$ with $[0,1]$ and let $V = [0,1]$. Further we let $(\xi_t)_{t\geq 0}$ be the Markov process with state space V^G and generator $\mathcal{G}$, where, for suitable $F : X \to \mathbb{R}$,

$$\mathcal{G}F(x) = \sum_{g,h\in G} a(g,h)[x(h) - x(g)]\frac{\partial F(x)}{\partial x(g)} + \sum_{g\in G} \varrho(x(g))\frac{\partial^2 F(x)}{\partial^2 x(g)}. \tag{3.13}$$

The migration kernel a is an irreducible random walk kernel on G, and the diffusion coefficient (or resampling function) ϱ is a function $\varrho : [0,1] \to [0,\infty)$ that satisfies

$$\begin{gathered} \varrho(0) = \varrho(1) = 0, \\ \varrho(r) > 0,\ r \in (0,1), \\ \varrho \text{ is Lipschitz continuous.} \end{gathered} \tag{3.14}$$

The ergodic theory of this process has been studied by Shiga (1980a,b) (for the case $\varrho(r) = r(1-r)$), Notahara and Shiga (1980) and Cox and Greven (1994). As with the voter model, there is either coexistence or local extinction of one type, depending on whether the symmetrized kernel $\widehat{a}$ defined in (3.1) is transient or recurrent. We assume here that $\widehat{a}$ is recurrent. Let $\Theta = [0,1]$, and for $\theta \in \Theta$ let $\mathcal{M}_\theta$ be the collection of $\mu \in \mathcal{P}(X)$ such that for all $g \in G$,

$$\lim_{s\to\infty} \int \mu(dx)(a_s x(g) - \theta)^2 = 0. \tag{3.15}$$

For $\theta \in \Theta$, let $\mu_\theta = (1-\theta)\delta_{\mathbf{0}} + \theta\delta_{\mathbf{1}}$, and note that $S_\theta = \{\mathbf{0},\mathbf{1}\}$. By Theorem 4 of Cox and Greven (1994), if $\mu \in \mathcal{M}_\theta$ then

$$\mathcal{L}^\mu[\xi_t] \overset{t\to\infty}{\Longrightarrow} (1-\theta)\delta_{\mathbf{0}} + \theta\delta_{\mathbf{1}}. \tag{3.16}$$

We prove here a recurrence result for (ξ_t) that extends a result of Fleischmann and Greven (1994) for a special G and a (see the proof of their Proposition 5.11).

Theorem 2 *Let $\theta \in (0,1)$ and $\mu \in \mathcal{M}_\theta$. Then, for all finite $H \subset G$,*

$$\mathbf{P}^\mu\left[\liminf_{t\to\infty} \sup_{h\in H} \xi_t(h) = 0 \textit{ and } \limsup_{t\to\infty} \inf_{h\in H} \xi_t(h) = 1\right] = 1. \tag{3.17}$$

Proof It suffices to verify that (A1) and (A2) hold, in which case $\mathcal{M}_\theta \subset \mathcal{D}(\mu_\theta)$ is in the domain of stochastic attraction of μ_θ and our conclusion is justified by Proposition 2.3. Fix $\mu \in \mathcal{M}_\theta$ and $t > 0$. To verify (A1) we must show that

$$\lim_{s\to\infty} \mathbf{E}^\mu[(a_s\xi_t(g) - \theta)(a_s\xi_t(h) - \theta)] = 0. \tag{3.18}$$

In order to compute the first and second moment we use Lemma 1 of Cox and Greven (1994):

$$\mathbf{E}^x[\xi_t(g)] = a_t x(g) \tag{3.19}$$

$$\mathbf{E}^x[\xi_t(g)\xi_t(h)] = a_t x(g) a_t x(h) + \sum_{l\in G} \int_0^t a_{t-r}(h,l)a_{t-r}(k,l)\mathbf{E}^x[\varrho(\xi_r(l))]\, dr. \tag{3.20}$$

Now it is straightforward to check the formula

$$\begin{aligned}&\mathbf{E}^{\mu}[(\xi_t(h)-\theta)(\xi_t(k)-\theta)]\\ &\qquad =\int \mu(dx)(a_tx(h)-\theta)(a_tx(k)-\theta)+\sum_l\int_0^t a_{t-r}(h,l)a_{t-r}(k,l)\mathbf{E}^{\mu}[\varrho(\xi_r(l))]\,dr. \end{aligned} \tag{3.21}$$

It follows that

$$\begin{aligned}&\mathbf{E}^{\mu}[(a_s\xi_t(g)-\theta)(a_s\xi_t(g)-\theta)]\\ &\qquad =\int \mu(dx)(a_{s+t}x(g)-\theta)^2+\sum_l\int_0^t a^2_{s+t-r}(g,l)\mathbf{E}^{\mu}[\varrho(\xi_r(l))]\,dr. \end{aligned} \tag{3.22}$$

The first term on the right side of (3.22) tends to 0 as $s\to\infty$ because $\mu\in\mathcal{M}_\theta$. The second term on the right side of (3.22) is bounded above by

$$\|\varrho\|_\infty\int_0^t \widehat{a}_{2(s+r)}(g,g)\,dr,$$

and this also tends to 0 as $s\to\infty$ (recall that $|G|=\infty$ and that a is irreducible, hence $\widehat{a}_r(g,g)\to\infty$ as $r\to\infty$). We have thus established (3.18)

In order to show that (A2) holds, it suffices to prove that for finite $H\subset G$ and $\varepsilon>0$,

$$\lim_{t\to\infty}\mu\Big(\{x:\ \mathbf{P}^x[\xi_t(h)<\varepsilon\ \forall h\in H]>1-\theta-\varepsilon \text{ and } \mathbf{P}^x[\xi_t(h)>1-\varepsilon\ \forall h\in H]>\theta-\varepsilon\}\Big)=1. \tag{3.23}$$

We break the proof of (3.23) into two parts. First, we show that for any $g\in G$ and $\varepsilon>0$,

$$\lim_{t\to\infty}\mu\Big(\{x:\ \mathbf{P}^x[\xi_t(g)<\varepsilon]>1-\theta-\varepsilon,\ \text{and } \mathbf{P}^x[\xi_t(g)>1-\varepsilon]>\theta-\varepsilon\}\Big)=1. \tag{3.24}$$

Then we show that for any $g,h\in G$ and $\varepsilon>0$,

$$\lim_{t\to\infty}\mu\big(\{x:\ \mathbf{P}^x[\,\big|\xi_t(g)-\xi_t(h)\big|>\varepsilon]>\varepsilon\}\big)=0. \tag{3.25}$$

It is easy to see that (3.24) and (3.25) imply (3.23)

Let $H\subset G$ be finite, let $\delta>0$, and define

$$\Gamma_t(\delta)=\big\{x:\big|a_tx(h)-\theta\big|<\delta \text{ for all } h\in H\big\}.$$

Since $\mu\in\mathcal{M}_\theta$ and H is finite, Chebyshev's inequality and (3.15) imply that that

$$\lim_{t\to\infty}\mu(\Gamma_t(\delta))=1. \tag{3.26}$$

Suppose now that $H=\{g,h\}$. In the proof of Theorem 4 in Cox and Greven (1994), it is shown that for $\delta>0$,

$$\mathbf{E}^x[(\xi_t(g)-\delta^2)(\xi_t(h)+\delta^2)]\geq a_tx(g)-q_t(\delta,g,h)-\delta^2, \tag{3.27}$$

where $q_t(\delta,g,h)\to 0$ as $t\to\infty$. (The quantity $q_t(\delta,g,h)$ is the probability that two random walks starting from g and h, which move independently according to the kernel a_s, and coalesce at rate c whenever they occupy the same site, coalesce by time t. The constant c depends on g, h, ϱ and δ, but is strictly positive.) After a little rearrangement (using (3.19)), this inequality implies that

$$0\leq \mathbf{E}^x[\xi_t(g)(1-\xi_t(h))]\leq q_t(\delta,g,h)+2\delta^2. \tag{3.28}$$

By choosing t large enough so that $q_t(\delta, g, h) < \delta^2$, we have

$$0 \le \mathbf{E}^x[\xi_t(g)(1-\xi_t(h))] \le 3\delta^2. \tag{3.29}$$

Setting $g = h$, Chebyshev's inequality implies

$$\mathbf{P}^x[\xi_t(g)(1-\xi_t(g)) \ge \delta] \le 3\delta. \tag{3.30}$$

Assume now that $0 < \delta < 1/4$. For $r \in [2\delta, 1-2\delta]$, $r(1-r) \ge \delta$. Therefore, for large t, the last estimate implies that

$$\mathbf{P}^x\big[\xi_t(g) \in [2\delta, 1-2\delta]\big] \le 3\delta. \tag{3.31}$$

Using (3.19) we get that for $x \in \Gamma_t(\delta)$, $\mathbf{E}^x[\xi_t(g)] = a_t x(g) \ge \theta - \delta$. On the other hand,

$$\mathbf{E}^x[\xi_t(g)] \le 2\delta\mathbf{P}^x[\xi_t(g) \le 2\delta] + \mathbf{P}^x[\xi_t(g) > 2\delta] = 1 - (1-2\delta)\mathbf{P}^x[\xi_t(g) \le 2\delta]. \tag{3.32}$$

On account of these estimates,

$$\mathbf{P}^x[\xi_t(g) < 2\delta] \le \frac{1-\theta+\delta}{1-2\delta}. \tag{3.33}$$

A similar argument gives the inequality

$$\mathbf{P}^x[\xi_t(g) > 1-2\delta] \le \frac{\theta+\delta}{1-2\delta}. \tag{3.34}$$

Combining (3.31), (3.33) and (3.34), we obtain that for $x \in \Gamma_t(\delta)$

$$\begin{aligned} \mathbf{P}^x[\xi_t(g) < 2\delta] &\ge 1 - 3\delta - \frac{\theta+\delta}{1-2\delta}\ , \\ \mathbf{P}^x[\xi_t(g) > 1-2\delta] &\ge 1 - 3\delta - \frac{1-\theta+\delta}{1-2\delta}. \end{aligned} \tag{3.35}$$

Given $\varepsilon > 0$, we may choose $\delta > 0$ small enough, and then t large enough so that $q_t(\delta, g, g) < \delta^2$, and for all $x \in \Gamma_t(\delta)$,

$$\mathbf{P}^x[\xi_t(g) < \varepsilon] > 1 - \theta - \varepsilon \quad \text{and} \quad \mathbf{P}^x[\xi_t(g) > 1-\varepsilon] > \theta - \varepsilon.$$

In view of (3.26), (3.24) holds.

To prove (3.25), suppose that $|\xi_t(g) - \xi_t(h)| > \delta$. Then, it must be the case that at least one of $\xi_t(g), \xi_t(h)$ belong to the interval $[\delta, 1-\delta]$, or, one of $\xi_t(g), \xi_t(h)$ is smaller than δ and the other larger than $1-\delta$. In the latter case, either $\xi_t(g)(1-\xi_t(h)) > \delta(1-\delta)$ or $\xi_t(h)(1-\xi_t(g)) > \delta(1-\delta)$. Therefore, $\mathbf{P}^x[|\xi_t(g) - \xi_t(h)| > \delta]$ is bounded above by

$$\begin{aligned} &\mathbf{P}^x\big[\xi_t(g) \in [\delta, 1-\delta]\big] + \mathbf{P}^x\big[\xi_t(h) \in [\delta, 1-\delta]\big] \\ &\quad + \mathbf{P}^x[\xi_t(g)(1-\xi_t(h)) > \delta(1-\delta)] + \mathbf{P}^x[\xi_t(h)(1-\xi_t(g)) > \delta(1-\delta)]. \end{aligned} \tag{3.36}$$

For t large enough so that $q_t(\delta, g, h) < \delta^2$, and all $x \in \Gamma_t(\delta)$, (3.29) and Chebyshev's inequality imply

$$\mathbf{P}^x[\xi_t(g)(1-\xi_t(h)) > \delta(1-\delta)] \le 3\delta/(1-\delta)$$

On account of this estimate, (3.31) and (3.36),

$$\mathbf{P}^x[|\xi_t(g) - \xi_t(h)| > \delta] < 6\delta + 6\delta/(1-\delta). \tag{3.37}$$

Given $\varepsilon > 0$, we may choose $\delta > 0$ small enough so that the right side above is less than ε, and t large enough so that $q_t(\delta, g, h) < \delta^2$. We therefore obtain that, for all $x \in \Gamma_t(\delta)$,

$$\mathbf{P}^x[|\xi_t(g) - \xi_t(h)| > \varepsilon] < \varepsilon. \tag{3.38}$$

In view of (3.26), (3.25) holds. □

3.3 Interacting Fleming Viot Processes

Here we consider a generalization of the two allele (A and B, say) model of the last example to infinitely many alleles. The space E of alleles (or types) is now infinite. W.l.o.g. we assume $E = [0, 1]$. The interval $[0, 1]$ is understood as an arbitrary labeling of the types. Though, we need some measurability of E and thus equip it with the Borel σ-field $\mathcal{B}$ from the Euclidean metric on $[0, 1]$.

Now $\xi_t(g)(A)$ is the frequency at time t of individuals in the colony $g \in G$ having a type that is in $A \in \mathcal{B}$. Hence $\xi_t(g) \in \Delta_E := V := \mathcal{P}(E, \mathcal{B})$ (the set of probability measures on $(E, \mathcal{B})$) and (ξ_t) is a Markov process with values in

$$X = \mathcal{P}(E, \mathcal{B})^G.$$

The process (ξ_t) is a model with migration and resampling. While the migration is just the one we introduced in the previous subsection we must be more careful with the resampling: we can define (ξ_t) uniquely only for the so-called Fisher-Wright case $\varrho(x) = c \cdot x(1 - x)$, $c > 0$.

We define (ξ_t) in terms of its generator $\mathcal{G}$ which is defined for certain polynomials $F : X \to \mathbb{R}$ by

$$\begin{aligned} \mathcal{G}F(x) \quad = \quad & \sum_{g,h \in G} a(g, h) \int_E \left(\frac{\partial F(x)}{\partial x(g)}(e) \right) (x(h)(de) - x(g)(de)) \\ & + \sum_{g \in G} \int_E \int_E \left(\frac{\partial^2 F(x)}{(\partial x(g))^2}(e, e') \right) \left[x(g)(de)\delta_e(de') - x(g)(de)x(g)(de') \right]. \end{aligned} \tag{3.39}$$

We do not explain the details of this formula but refer to Dawson, Greven and Vaillancourt (1995), equation (0.8), or Chapter 2.6 of Dawson (1993).

The possible limit points μ_θ will now be indexed by the set $\Theta = \mathcal{P}(E, \mathcal{B})$. We define for $\theta \in \Theta$

$$\mathcal{M}_\theta = \left\{ \mu \in \mathcal{P}(\mathcal{P}(E, \mathcal{B})^G) : \lim_{s \to \infty} \int \mu(dx)(a_s x(A) - \theta(A))^2 = 0 \ \forall A \in \mathcal{B} \right\}. \tag{3.40}$$

Again for $\widehat{a}$ recurrent we have clustering and the limit points are mixtures of the measures $\delta_{\boldsymbol{\delta_e}}$ (the bold symbol indicates the point mass on the constant state $\boldsymbol{\delta_e}$, $e \in E$):

$$\mu_\theta = \int_E \theta(de)\ \delta_{\boldsymbol{\delta_e}}. \tag{3.41}$$

See Dawson, Greven and Vaillancourt (1995), Theorem 0.1. In particular, $S_\theta = \operatorname{supp}(\mu_\theta) = \{\boldsymbol{\delta_e} : \ e \in \operatorname{supp}(\theta)\}$.

Theorem 3 *Let $\theta \in \Theta$ and $\mu \in \mathcal{M}_\theta$. Then for every finite set $H \subset G$ and every set $A \in \mathcal{B}$ with $\mu_\theta(A) > 0$,*

$$\mathbf{P}^\mu \left[\limsup_{t \to \infty} \inf_{h \in H} \xi_t(h)(A) = 1 \right] = 1. \tag{3.42}$$

In particular, the locally predominant type changes infinitely often.

Proof For fixed $A \in \mathcal{B}$ the process $(\widetilde{\xi}_t(g);\ g \in G)_{t \geq 0} = (\xi_t(g)(A);\ g \in G)_{t \geq 0}$ is just the process of interacting Fisher-Wright diffusions on $[0, 1]$. That is the process of interacting diffusions from the last example with diffusion coefficient $\varrho(x) = x(1 - x)$. Hence the claim follows from Theorem 2. □

3.4 Interacting Brownian motions

So far we have considered examples where the state space (at each site) was compact. Now we come up with our first example of a non-compact state space.

Here we consider only one type, i.e., $E = \{1\}$. In the notation of the last few examples we have $\Theta = V = \mathbb{R}$ and $X \subset \mathbb{R}^G$ is a Liggett–Spitzer space (see Liggett and Spitzer (1981)). More precisely, fix $\gamma \in (0,\infty)^G$ with $\sum_{g\in G}\gamma(g) < \infty$ and with the property that

$$\sup_{g\in G}\gamma(g)^{-1}\,(\gamma a)(g) < \infty. \tag{3.43}$$

Now define $\|x\|_\gamma = \sum_{g\in G}|x(g)|\gamma(g)$ and let

$$X = \big\{x \in \mathbb{R}^G :\ \|x\|_\gamma < \infty\big\}. \tag{3.44}$$

For example, if $G = \mathbb{Z}^d$ and a is the kernel of simple random walk then $\gamma = (1 + \|x\|_2)^{-p}$ fulfills the above assumption for $p > d$. Hence all $x \in \mathbb{R}^{\mathbb{Z}^d}$ that do not grow faster than a polynomial are possible initial configurations.

We define linearly interacting Brownian motions as the Markov process on X with generator

$$\mathcal{G}F(x) = \sum_{g,h\in G} a(g,h)\,[x(h) - x(g)]\,\frac{\partial F(x)}{\partial x(g)} + \frac{1}{2}\sum_{g\in G}\frac{\partial^2 F(x)}{\partial x(g)^2}. \tag{3.45}$$

Note that $\mu \circ ((a_t x)(g))^{-1}$ is the distribution of the random variable $a_t x(g)$ if x is distributed according to $\mu(dx)$. Define

$$\mathcal{M} = \Big\{\mu \in \mathcal{P}(X) :\ \{\mu \circ ((a_t x)(g))^{-1},\ t \geq 0\} \text{ is tight } \forall\, g \in G\Big\}. \tag{3.46}$$

We show that if the symmetrized kernel $\widehat{a}$ is recurrent, then for $\mu \in \mathcal{M}$

$$\mathcal{L}^\mu[\xi_t] \overset{t\to\infty}{\Longrightarrow} \mu_0 := \frac{1}{2}\delta_{-\boldsymbol{\infty}} + \frac{1}{2}\delta_{+\boldsymbol{\infty}}, \tag{3.47}$$

and moreover that $\mathcal{M} \subset \mathcal{D}(\mu_0)$, the domain of stochastic attraction of μ_0 (Of course, other subsets of $\mathcal{D}(\mu_0)$ are conceivable). To make precise sense of this statement let $\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ be the two point compactification of the real line. The bold symbols $-\boldsymbol{\infty}$ and $+\boldsymbol{\infty}$ denote the elements in $\bar{\mathbb{R}}^G$ with all components equal to $-\infty$ respectively $+\infty$.

Note that (3.47) is "convergence of the means" in the sense of (2.2). Here even the stronger statement of stochastic convergence needed for (A2) is true

$$\mathcal{L}^x[\xi_t] \overset{t\to\infty}{\Longrightarrow} \mu_0, \qquad \mu(dx)\text{-stochastically} \tag{3.48}$$

or equivalently: for all $\varepsilon > 0$, $K > 0$ and $H \subset G$ finite

$$\liminf_{t\to\infty}\mu\Big(\Big\{x :\ \mathbf{P}^x\Big[\inf_{h\in H}\xi_t(h) > K\Big] \bigwedge \mathbf{P}^x\Big[\sup_{h\in H}\xi_t(h) < -K\Big] > \frac{1}{2} - \varepsilon\Big\}\Big) = 1. \tag{3.49}$$

We give the simple proof of (3.49): First note that $(\xi_t)_{t\geq 0}$ solves a system of stochastic differential equations

$$d\xi_t(g) = \sum_{h\in G} a(g,h)[\xi_t(h) - \xi_t(g)]\,dt\ + dW_t(g), \tag{3.50}$$

where $\{(W_t(g)_{t\geq 0},\ g \in G\}$ is an independent family of standard Wiener processes. (This can be seen by an approximation procedure as in Shiga and Shimizu (1980), proof of Theorem 3.2.) Hence ξ_t can be written as

$$\xi_t(g) = (a_t\xi_0)(g) + \int_0^t \sum_{h\in G} a_{t-s}(g,h)\,dW_s(h). \tag{3.51}$$

¿From (3.51) we derive for $x \in X$ the first and second moment:

$$\begin{aligned} \mathbf{E}^x[\xi_t(g)] &= (a_t x)(g), & (3.52) \\ \mathbf{Cov}^x[\xi_t(g), \xi_t(h)] &= \frac{1}{2}\widehat{G}_{2t}(g,h), & (3.53) \end{aligned}$$

where $\widehat{G}_t(g,h)$ is the Green function of the symmetrized kernel $\widehat{a}$

$$\begin{aligned} \widehat{G}_t(g,h) &= \int_0^t \widehat{a}_s(g,h)\, ds \\ &= \int_0^t \sum_{l \in G} a_{s/2}(g,l) a_{s/2}(h,l)\, ds. \end{aligned}$$

Since a is irreducible and $\widehat{a}$ is recurrent, the weak ratio limit theorem (see, e.g., Spitzer's book, Proposition 1.5) implies

$$\frac{\widehat{G}_t(g,h)}{\widehat{G}_t(g,g)} \overset{t\to\infty}{\longrightarrow} 1. \qquad (3.54)$$

Hence asymptotically the components are perfectly correlated while

$$\mathbf{Var}^x[\xi_t(g)] = \frac{1}{2}\widehat{G}_{2t}(0,0) \overset{t\to\infty}{\longrightarrow} \infty. \qquad (3.55)$$

Since under $\mathbf{P}^x$ the field $\{\xi_t(g),\ g \in G\}$ is Gaussian and $\mathbf{E}^x[\xi_t(g)] = (a_t x)(g)$ is tight as $t \to \infty$ (w.r.t. μ) for all $g \in G$, this implies (3.49). Hence we have shown that (A2) holds.

Assumption (A1) however is an immediate consequence of (3.51). Thus we can apply Proposition 2.3 to get the following result:

Theorem 4 *Let $\mu \in \mathcal{M}$. Then for $H \subset G$ finite*

$$\mu\Big(\Big\{x:\ \mathbf{P}^x\Big[\limsup_{t\to\infty} \inf_{h\in H} \xi_t(h) = \infty,\ \liminf_{t\to\infty} \sup_{h\in H} \xi_t(h) = -\infty\Big] = 1\Big\}\Big). \qquad (3.56)$$

□

3.5 Mutually catalytic branching super random walk

We now come to the example that mainly motivated our work. Consider a two-type "infinitesimal mass" interacting particle system on $\mathbb{Z}^d$, i.e. $G = \mathbb{Z}^d$, $E = \{1,2\}$, $V = [0,\infty)^2$. Hence $\xi_t(i)(c) \in [0,\infty)$ is the amount of mass of type $c \in \{1,2\}$ at site $i \in \mathbb{Z}^d$ at time $t \geq 0$. The particles migrate (independently of each other) according to a nearest neighbor random walk, i.e. $a(i,j) = \frac{1}{2d}\mathbb{1}_{\{|i-j|=1\}}$. Additionally the mass of each type fluctuates randomly according to Feller's branching diffusion, however with a diffusion rate proportional to the mass of the other type at that particular site. The proper space of the process is a subspace $X \subset V^G$ that fulfills a natural growth condition (see Theorem 1.1 of Dawson and Perkins (1998)).

Formally we define $(\xi_t)_{t\geq 0}$ as the Markov process on X with generator $\mathcal{G}$ given by

$$\mathcal{G}F(x) = \sum_{c=1}^{2} \Big[\sum_{i,j\in\mathbb{Z}^d} a(i,j)[x(j)(c) - x(i)(c)] \frac{\partial F(x)}{\partial x(i)(c)} + \sum_{i\in\mathbb{Z}^d} x(i)(c)x(i)(3-c) \frac{\partial^2 F(x)}{\partial x(i)(c)^2} \Big]. \qquad (3.57)$$

The explicit construction of this process can be found in Dawson and Perkins (1998). Uniqueness in law is based on Mytnik's duality (see Mytnik (1996)).

Dawson and Perkins investigate the longtime behavior of (ξ_t). They show that if $d = 1$ or $d = 2$ and $\xi_0 \equiv \theta \in (0,\infty)^2$ then locally one type dies out (in probability) while the other type is locally constant but random. The question that was raised by Ed Perkins at the 1997 Vancouver Probability Meeting is whether it is always (i.e. as time passes) the same type that is locally predominant. From the above discussion the reader might by now guess the right answer. Here however, we first want to give the result of Dawson and Perkins in detail.

Consider planar Brownian motion $(B_t)_{t\geq 0}$ in the upper right quadrant V, started in $\theta \in (0,\infty)^2$. Define m_θ to be the distribution of the first hitting of B_t of the boundary $\partial V = \{0\} \times [0,\infty) \cup [0,\infty) \times \{0\}$ of V. m_θ is absolutely continuous w.r.t. Lebesgue measure on ∂V and $\mathrm{supp}(m_\theta) = \partial V$ for $\theta \in (0,\infty)^2$. In fact, it is an exercise to compute the density of m_θ. Letting

$$g(x,y,z) = \frac{4xy}{\pi}\Big(4x^2y^2 + (z^2+y^2-x^2)^2\Big)^{-1} z, \qquad x,y,z \geq 0,$$

we have

$$\begin{aligned} m_\theta(dv_1,\{0\}) &= g(\theta(1),\theta(2),v_1)dv_1, \\ m_\theta(\{0\},dv_2) &= g(\theta(2),\theta(1),v_2)dv_2. \end{aligned}$$

Further let $\delta_{\mathbf{v}}$ be the unit mass at the element $\mathbf{v} \in X$ with all components equal to $v \in V$. Finally define

$$\begin{aligned} \mu_\theta &= \int_E m_\theta(dv)\delta_{\mathbf{v}} && (3.58) \\ &= \int_{[0,\infty)} m_\theta(dv_1,\{0\})\delta_{(\mathbf{v_1},\mathbf{0})} + \int_{[0,\infty)} m_\theta(\{0\},dv_2)\delta_{(\mathbf{0},\mathbf{v_2})}. && (3.59) \end{aligned}$$

Theorem 1.5 of Dawson and Perkins (1998) says that for the constant state $\boldsymbol{\theta} \in X$ with all components equal to θ,

$$\mathcal{L}^{\boldsymbol{\theta}}[\xi_t] \overset{t\to\infty}{\Longrightarrow} \mu_\theta. \tag{3.60}$$

In order to apply our abstract argument we have to have an invariant class $\mathcal{M}_\theta \subset \mathcal{D}(\mu_\theta)$ in the domain of stochastic attraction of μ_θ. A large class $\mathcal{M}_\theta$ with these features has been obtained by Cox, Klenke and Perkins (1999). They show in their Theorem 2 that

$$\mathcal{M}_\theta = \left\{\mu \in \mathcal{P}(X) :\ C_\mu < \infty,\ \lim_{t\to\infty} \int \mu(dx)\ (a_t x(i)(c) - \theta(c))^2 = 0,\ i \in \mathbb{Z}^d,\ c = 1,2\right\}, \tag{3.61}$$

where

$$C_\mu := \sup_{i\in\mathbb{Z}^d} \int \mu(dx)\ x(1)(i)^2 + x(2)(i)^2, \tag{3.62}$$

ensures convergence in the sense of (A2). In particular, we have

$$\mathcal{L}^{\mu}[\xi_t] \overset{t\to\infty}{\Longrightarrow} \mu_\theta, \qquad \mu \in \mathcal{M}_\theta,\ \theta \in \Theta. \tag{3.63}$$

It is simple to check invariance of $\mathcal{M}_\theta$ (A1). In fact, for all $T > 0$, $i \in \mathbb{Z}^d$ and $c = 1,2$, by Theorem 2.2 of Dawson and Perkins (1998),

$$\begin{aligned} \mathbf{E}^\mu[\xi_T(i)(c)^2] &= \mathbf{E}^\mu[(a_T\xi_0(c))(i)^2] && (3.64) \\ &\quad + \int_0^T dr \sum_{k\in\mathbb{Z}^d} \mathbf{E}^\mu[a_{T-r}(k,i)^2 a_r\xi_0(k)(1) a_r\xi_0(k)(2)]. \end{aligned}$$

Hence Jensen's inequality yields (with $G_T = \int_0^T a_t(0,0)dt$)

$$C_{\mu\mathcal{S}(T)} \leq (1 + G_{2T})C_\mu < \infty. \tag{3.65}$$

On the other hand, again by Theorem 2.2 of Dawson and Perkins,

$$\begin{aligned}\mathbf{E}^{\mu}[(a_{t-T}\xi_T(i)(c)-\theta(c))^2]-\int\mu(dx)(a_tx)(i)(c)-\theta(c))^2 \\ = \int_0^T dr\sum_{k\in\mathbb{Z}^d}\mathbf{E}^{\mu}[a_{t-r}(k,i)^2a_r\xi_0(k)(1)a_r\xi_0(k)(2)] \\ \leq \frac{1}{2}C_{\mu}\cdot(G_{2t}-G_{2(t-T)})\overset{t\to\infty}{\longrightarrow}0.\end{aligned} \tag{3.66}$$

Hence $\mu\mathcal{S}(T)\in\mathcal{M}_\theta$ and $\mathcal{M}_\theta$ fulfills assumption (A1).

Now we present the main new result of this work which is an immediate consequence of Proposition 2.3 and Cox, Klenke and Perkins (1999).

Theorem 5 *Fix $\theta\in(0,\infty)$ and $\mu\in\mathcal{M}_\theta$. For every finite set $H\subset\mathbb{Z}^d$ and every $\rho\in\{0\}\times[0,\infty)\cup[0,\infty)\times\{0\}$*

$$\mathbf{P}^{\mu}\left[\liminf_{t\to\infty}\sup_{i\in H}\|\xi_t(i)-\rho\|=0\right]=1. \tag{3.67}$$

In particular, $\mathbf{P}^{\mu}$-a.s. the locally predominant type changes infinitely often.

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