Catalytic Branching and the Brownian Snake

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Abstract

We construct a catalytic super process X (measure-valued spatial branching process) where the local branching rate is governed by an additive functional A of the motion process. These processes have been investigated before but under restrictive assumptions on A. Here we do not even need continuity of A. The key is to introduce a new time scale in which motion and branching occur at a varying speed but are continuous.

Another aspect is to consider X in the generic time scale of the branching - and not of the motion process. This allows to give an explicit construction of X using the Brownian snake. As a by-product this yields an almost sure approximation by the corresponding branching particle systems.

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1 Introduction

1.1 Motivation

Consider a spatial branching particle system where particles move (independently) according to some Markov process ξ in $\mathbb{R}^d$. After an exponential lifetime a particle either splits into two particles or dies - either choice with probability $\frac{1}{2}$. The offspring particles follow the same dynamics. It is well known that one can perform a diffusion limit (of small particle masses and short lifetimes) to obtain a measure-valued branching process. These so-called super-processes are a well-studied object and we only briefly refer to Dawson [Daw93] or Etheridge [Eth00] for reference.

Picking up a classical idea, in the last years there has been growing interest in a situation where the *rate* at which the branching occurs varies in time and space. The very basic idea is that the rate of the branching is proportional to the concentration of some (hypothetical) matter that catalyzes the process of branching. Thus these processes in a space-time varying (maybe random) medium are known as catalytic spatial branching processes. See Dawson and Fleischmann [DF00] or [Kle00] for an overview on catalytic branching.

A slightly different point of view is the following: Once a particle is born it gets an individual exponentially distributed lifetime with a fixed mean. Furthermore each particle has an individual clock A that runs at a varying speed (governed by the medium). When the clock reaches the lifetime the particle dies or splits. Technically, this clock A is an increasing additive functional of the motion process ξ of the particle.

A situation where each (infinitesimal) particle has its own independent clock which follows a time-homogeneous increasing Lévy process (so-called branching with subordination, see [BLGLJ97]) has been considered by several authors. By expanding the state space of the motion process one can consider branching with subordination as a special case of catalytic branching, though with a clock that is too irregular (in particular discontinuous) to fit in the general framework of catalytic branching considered so far. We will see in this paper that the assumptions on the clock can be relaxed so that branching with subordination indeed fits into the framework of catalytic branching.

So far one had to make strong assumptions on A in order to construct the catalytic super process (see Dynkin [Dyn94, Chapter 3.2], Dawson [Daw93, Chapter 4.2], or Dawson and Fleischmann [DF97]). In particular, continuity of A was needed. Here we can drop all assumptions on A by taking a different point of view on the process. The means is to introduce in a first step an auxiliary process in a new time scale $\gamma(t) = t + A(t)$. This is the time a computer would need to move and branch a particle. In this new time scale particles move and branch at a varying speed. However, both t and $A(t)$ are Lipschitz continuous functions of γ and hence we are in the framework of [Dyn94] and [Daw93]. Note that in this time scale, the particle simply stands still for a while if A makes a jump.

In a second step one has to make the inverse time-change to get back the original model. This time-change is a bit involved since each particle needs a different time-change. We will see that this problem can be solved using exit measures and the so-called special Markov property. Note that if A makes a jump, after the inverse time-change we can observe the process only *after* that jump. If also the motion has a point of discontinuity the particles first make the spatial jump and then use their branching time. We can observe only the random number of particles at the destination of the spatial jump after the branching. There is no choice for the order in which these things happen as long as one wants a cadlag process.

A third aspect (after the motion's generic time scale t and the universal time scale γ) is to consider the generic time of the branching process as the time scale. Thus we get particles that branch at constant rate one but *move* at a varying speed. Again a discontinuity of A simply means that the particle stands still for a while. Also, in order to obtain a cadlag process, we have to arrange things in such a way that in cases of ambiguities particles first move and then branch. This time scale allows to give an explicit pathwise construction of the time-changed process by means of Le Gall's Brownian snake process (see [LG91, LG94, LG99]). In this approach one first samples the complete genealogy of all particles for all times and then adapts the motion processes to this genealogy. As a spin-off one gets a nice almost sure approximation result for the catalytic super process by the corresponding particle systems with short life times.

Note that our method differs from approaches to time changed catalytic branching that have been made before. Fleischmann and Le Gall [FLG95] obtain a nice description of one-dimensional super Brownian motion with a single point catalyst in terms of a subordinator and excursions from the catalytic point. Delmas [Del96] has generalized this to a higher dimensional situation. Bertoin, Le Jan and Le Gall [BLGLJ97] consider a situation where A does not depend on the spatial motion and is a subordinator. They also assume that the motion process in the branching time scale is continuous (assumption (H) on page 43 of [BLGLJ97]). This assumption allows them to work with a topology in which there exists a continuous version of the snake. However without that assumption the snake need not even to be measurable (see the example below our Theorem 3).

A very nice representation of a catalytic branching process in terms of a modified snake process is due to Dhersin and Serlet [DS00]. In their description the lifetime process of the snake (which encodes the genealogy) is not reflected Brownian motion but a diffusion process with diffusion coefficient depending on the current endpoint of the spatial paths (the positions the infinitesimal particles). In particular, the genealogy cannot be described autonomously before knowing the spatial motion. The approach of [DS00] is based on a stochastic calculus treatment of the snake process. In terms of regularity of the clock it requires A to be continuously differentiable with the derivative bounded and bounded away from 0.

Our description of catalytic super-Brownian motion in terms of the Brownian snake is more similar to the construction given in [DFM01]. However, they consider only a very special catalyst and a special motion process such that the motion process in the branching time scale is continuous. In particular, their catalyst is not allowed to have gaps. Here we aim at the most general situation where the catalyst is allowed to be virtually anything. Also the construction and almost sure convergence of the embedded particle system is new.

1.2 Background

We start by collecting the ingredients of the processes to be constructed. Let $(\xi_t)_{t\geq 0}$ be a (possibly time-inhomogeneous) Feller process with values in a locally compact Polish space E. By $(\mathbf{P}_{s,x},\ s\geq 0,\ x\in E)$ we denote the family of probability measures on the Skorohod space $D(\mathbb{R}^+, E)$ associated with the process ξ started at time s in the point x (and with the convention that $\xi_r = x$ for $r\leq s$). We will use the notation $x_{\leq t}$ to denote the path up to time t that is constant at $x\in E$. Let $C_b(E)$ $[C_b^+(E)]$ denote the space of [nonnegative] bounded continuous functions on E and let $\mathcal{M}_f(E)$ denote the space of finite Borel measures on E. For $f\in C_b(E)$ and $\mu\in\mathcal{M}_f(E)$ we write $\langle \mu, f\rangle := \int f\,d\mu$. We fix a complete metric d on E and for an interval I we denote by $D(I,E)$ the Skorohod space of cadlag functions $I\to E$.

We assume that A is an increasing additive functional of ξ. That is, $A(\xi, dt)$ is an adapted locally finite measure on $\mathbb{R}^+$: For every $s\leq t$, $A(\xi,(s,t])$ is measurable with respect to $\sigma((\xi_r)_{r\in(s,t]})$. A is the clock that governs the branching of a particle following the path ξ. We will assume $A(\{0\}) = 0$ and use the notation $A(t) := A_\xi(t) = A(\xi,(0,t])$ for the time the clock shows if initially set at zero.

For the moment let us assume that A is a so-called "branching functional". Essentially this means that A is continuous and fulfills certain moment conditions (see Dynkin [Dyn94]). We will later only need that A is a branching functional if $t\mapsto A_\xi(t)$ is Lipschitz continuous (with constant 1). For branching functionals one can construct catalytic spatial branching processes X^h, $h>0$, with the following rules

- New particles independently get assigned lifetimes that are exponentially distributed with parameter $1/h$.
- Particles move independently according to the ξ–motion.
- Assume that a particle is born at time s and has lifetime λ. At the time t when the individual (branching) age $A((s,t])$ first exceeds λ, the particle performs a critical binary branching event.

For $B\subset E$ Borel we write

$$X_t^h(B) = \#\{\text{particles in } B \text{ at time } t\}. \tag{1.1}$$

Thus X^h is a Markov process with values in $\mathcal{M}_f(E)$.

If we perform the limit of high densities and short lifetimes, i.e., if we let $h\to 0$ (and assume that the initial conditions hX_0^h converge) then hX^h converges to a so–called catalytic super process X with values in $\mathcal{M}_f(E)$ (see, e.g., [Daw93] or [Dyn93, Thm. I.3.1]). This X is a multiplicative Markov process which can be characterized by its log-Laplace transforms

$$u(s,t,\varphi;\,x) := -\log \mathbf{E}_{s,\delta_x}\left[\exp(-\langle X_t,\varphi\rangle)\right], \qquad s\leq t. \tag{1.2}$$

Here $\varphi \in C_b^+(E)$ is a test function and $\mathbf{E}_{s,\delta_x}$ denotes the expectation when $X_s = \delta_x$ almost surely. (We will use the symbol $\mathbf{P}_{s,\delta_x}$ for the corresponding probability.) In fact, multiplicativity means that for $\mu \in \mathcal{M}_f(E)$

$$-\log \mathbf{E}_{s,\mu}\left[\exp(-\langle X_t, \varphi\rangle)\right] = \langle \mu, u(s,t,\varphi)\rangle. \tag{1.3}$$

Under the above conditions on A, u is the unique non-negative solution of the cumulant equation

$$u(s,t,\varphi;x) = \mathbf{E}_{s,x}\left[\varphi(\xi_t) - \int_s^t A(\xi, dr)u^2(r,t,\varphi;\xi_r)\right]. \tag{1.4}$$

Note that in this setting one cannot drop the assumption that $r \mapsto A(\xi,(s,r])$ is a continuous function (else there might be no solution of (1.4)). The approach of Dawson and Fleischmann [DF97] is to show that (1.4) can be solved uniquely if A is a "nice branching functional" and then to construct X by means of Kolmogorov's extension theorem.

The same equation (1.4) occurs as Kolmogorov's backward equation for the Laplace transforms of the particle system X^h. For $\mu \in \mathcal{M}_f(E)$ denote by $\mathcal{H}(\mu)$ the probability measure on $\mathcal{M}_f(E)$ which is the law of a Poisson point process on E with intensity measure μ. Further let $\mathbf{P}_{s,\mathcal{H}(\mu)}$ and $\mathbf{E}_{s,\mathcal{H}(\mu)}$ denote probability and expectation of X^h if the initial state X_s^h has distribution $\mathcal{H}(\mu)$. Then

$$-\log\left(\mathbf{E}_{s,\mathcal{H}(\mu)}\left[\exp(-\langle X_t^h, \varphi\rangle)\right]\right) = \langle \mu, u_h(s,t,1-e^{-\varphi})\rangle, \tag{1.5}$$

where

$$u_h(s,t,\varphi;x) := h\,u(s,t,\varphi/h). \tag{1.6}$$

Note that, formally at least, $h^{-1}u_h(s,t,1-e^{-h\varphi}) \to u(s,t,\varphi)$ as $h \to 0$, which means just that X is the limit of hX^h.

1.3 The Universal Time Scale

In this section we assume only that A is an increasing additive functional of ξ but do not impose additional assumptions.

Both the motion's and the branching's generic time scales have one disadvantage: They might result in discontinuous behavior of the complementary mechanism. If A has constant intervals, then the motion process has jumps if seen from the branching time scale. If A has jumps, then there is a discontinuity in the branching if viewed from the motion's time scale.

The Time Change

The easy way out comes in sight if one thinks of how to program the model on a computer. If motion and branching consume the same amount of CPU time, the amount of time needed to simulate an individual particle up to time t is

$$\gamma(t) = t + A(t). \tag{1.7}$$

If we denote by $t(\gamma)$ the time the computer uses to move the particle and by $T(\gamma)$ the time the computer uses to branch it, then we have

$$\begin{aligned} t(\gamma) &= \inf\big\{s:\ A(s)+s>\gamma\big\} \\ T(\gamma) &= \gamma - t(\gamma). \end{aligned} \tag{1.8}$$

Hence if A is continuous in the point $t(\gamma)$, then

$$\gamma = t(\gamma) + T(\gamma) = t(\gamma) + A(t(\gamma)). \tag{1.9}$$

Note that both $\gamma \mapsto t(\gamma)$ and $\gamma \mapsto T(\gamma)$ are Lipschitz continuous with constant 1.

Define

$$\zeta_\gamma = (\zeta_\gamma^\xi, \zeta_\gamma^t, \zeta_\gamma^\Delta), \tag{1.10}$$

where

$$\begin{aligned} \zeta_\gamma^\xi &= \xi_{t(\gamma)} \\ \zeta_\gamma^t &= t(\gamma) \\ \zeta_\gamma^\Delta &= A(t(\gamma)) - T(\gamma). \end{aligned} \tag{1.11}$$

The third coordinate keeps track of the remaining branching time if A has a jump. That is, $t(\gamma) = t(\gamma + \zeta_\gamma^\Delta)$ and $t(\gamma) < t(\gamma')$ for $\gamma' > \gamma + \zeta_\gamma^\Delta$. This book keeping is necessary to make ζ a Markov process. In fact, $(\zeta_\gamma)_{\gamma\geq 0}$ is a time-homogeneous Markov process.

Clearly $\gamma \mapsto \gamma - t(\gamma)$ is the distribution function of a "nice branching functional". We denote by $T'(\gamma) = 1 - \frac{dt(\gamma)}{d\gamma}$ its derivative. Thus we can use the theory of [Daw93, Chapter 4.2] or Dynkin [Dyn94, Chapter 3] to define the corresponding catalytic super process $(Z_\gamma)_{\gamma\geq 0}$ and the corresponding particle system $(Z_\gamma^h)_{\gamma\geq 0}$. The processes Z and Z^h are also time-homogeneous and the log-Laplace transform $u_Z(\gamma_1, \gamma_2, \varphi) = u_Z(\gamma_2 - \gamma_1, \varphi)$ is the unique non-negative solution of

$$u_Z(\varrho, \varphi; (x,t,\Delta)) = \mathbf{E}_{(x,t,\Delta)}\left[\varphi(\zeta_\varrho) - \int_0^\varrho d\gamma\, T'(\gamma)\, u_Z^2(\varrho-\gamma, \varphi; \zeta_\gamma)\right]. \tag{1.12}$$

By the semigroup property of u and the fact that $\zeta_\gamma = (\zeta_0^\xi, \zeta_0^t, \zeta_0^\Delta - \gamma)$ for $\gamma \in [0, \zeta_0^\Delta)$ we get

$$u_Z(\varrho+\Delta, \varphi; (x,t,\Delta)) = u_Z(\varrho, \varphi; (x,t,0)) - \int_0^\Delta dr\, u_Z^2(\varrho+\Delta-r, \varphi; (x,t,\Delta-r)). \tag{1.13}$$

This integral equation (with Δ as variable) can be solved explicitly:

$$u_Z(\varrho+\Delta, \varphi; (x,t,\Delta)) = (\Delta + u_Z^{-1}(\varrho, \varphi; (x,t,0)))^{-1} \tag{1.14}$$

where $(\Delta + u^{-1})^{-1}$ is understood to be 0 if $u = 0$.

The Inverse Time Change

Having constructed these branching processes we want to get back the catalytic branching processes (in the motion's time scale) that we were originally interested in. To this end we have to make the inverse time transformation on the level of the individual particles. Technically this can be done by introducing the exit measures for particles stopped when their motion time exceeds a given value t.

We introduce the stopping time for ζ

$$\tau_t := \inf\big\{\gamma : \zeta^t_\gamma > t\big\}. \tag{1.15}$$

Note that $\tau_t = A(t)+t$ and that $\zeta_{\tau_t} \overset{d}{=} (\xi_t, t, 0)$. Now we can build the *exit measures* $Z^h_{\tau_t}$ and Z_{τ_t}. For a rigorous and extensive treatment see, e.g., [Dyn93]. Heuristically, the random measure $Z^h_{\tau_t} \in \mathcal{M}_f((E\times(\mathbb{R}^+)^2))$ is obtained by stopping the particles of Z^h individually when the motion time $t(\gamma)$ exceeds the value t. Stopped particles neither move nor branch. The measure $Z^h_{\tau_t}$ keeps track of the points ζ_{τ_t} where the particles get stopped. (This is the usual way to define the exit measures in the time-homogeneous setting.) For the exit measure Z_{τ_t} one has the same heuristics for infinitesimal particles.

More formally the exit measures can be characterized by their log-Laplace transforms

$$u_{Z,t}(\varphi;(x,s,\Delta)) := -\log \mathbf{E}_{(x,s,\Delta)}\left[e^{-\langle Z_{\tau_t},\varphi\rangle}\right]. \tag{1.16}$$

(for $\varphi \in C^+_b(E\times(\mathbb{R}^+)^2)$) which are the unique nonnegative solutions of the integral equation

$$u_{Z,t}(\varphi;(x,s,\Delta)) = \mathbf{E}_{(x,s,\Delta)}\left[\varphi(\zeta_{\tau_t}) - \int_0^{\tau_t} d\gamma\, T'(\gamma)u^2_{Z,t}(\varphi;\zeta_\gamma)\right]. \tag{1.17}$$

Multiplicativity of Z_{τ_t} is tantamount to

$$\langle \mu, u_{Z,t}(\varphi)\rangle = -\log \mathbf{E}_\mu\left[e^{-\langle Z_{\tau_t},\varphi\rangle}\right], \qquad \mu\in\mathcal{M}_f(E\times(\mathbb{R}^+)^2). \tag{1.18}$$

Though also for the particle systems the formulas for the log-Laplace transforms of the exit measures are not difficult we give it here only for the case of Poisson initial data $\mathcal{H}(\mu)$ with intensity measure μ as they take then a particularly appealing form (see [Dyn93, Section I.2.4])

$$-\log \mathbf{E}_{\mathcal{H}(\mu)}\left[e^{-\langle Z_{\tau_t},\varphi\rangle}\right] = \langle \mu, u_{Z^h,t}(1-e^{-\varphi})\rangle, \tag{1.19}$$

where (compare (1.6))

$$u_{Z^h,t}(\varphi) = h u_{Z,t}(\varphi/h). \tag{1.20}$$

Again, since $\zeta_\gamma = (\zeta^\xi_0, \zeta^t_0, \zeta^\Delta_0 - \gamma)$ for $\gamma\in[0,\zeta^\Delta_0)$ we see that

$$\begin{aligned} &u_{Z,t}(\varphi;(x,s,\Delta))\\ &= \mathbf{E}_{(x,s,0)}\Big[\tilde\varphi(\zeta_{\tau_t}) - \int_0^{\tau_t} d\gamma\, T'(\gamma)u^2_{Z,t}(\varphi;\zeta_\gamma)\Big] - \int_0^{\Delta} u^2_{Z,t}(\varphi;(x,s,\gamma))d\gamma \end{aligned} \tag{1.21}$$

with the unique solution (compare (1.14))

$$u_{Z,t}(\varphi;(x,s,\Delta)) = \Big(\Delta + u_{Z,t}^{-1}(\varphi;(x,s,0))\Big)^{-1}. \tag{1.22}$$

As a consequence of Dynkin's so-called *special Markov property* (see [Dyn91, Theorem 1.5] or [Dyn93, Theorem I.1.3 on page 1195]) we have

$$(Z_{\tau_t})_{t\ge 0} \quad \text{and} \quad (Z^h_{\tau_t})_{t\ge 0} \quad \text{are Markov processes.} \tag{1.23}$$

Now we can define for $B \subset E$ Borel

$$\begin{aligned} X_t(B) &= Z_{\tau_t}(B\times(\mathbb{R}^+)^2) = Z_{\tau_t}(B\times\{t\}\times\{0\}),\\ X^h_t(B) &= Z^h_{\tau_t}(B\times(\mathbb{R}^+)^2) = Z^h_{\tau_t}(B\times\{t\}\times\{0\}). \end{aligned} \tag{1.24}$$

These measures keep track only of the spatial distribution of the stopped particles. In general, projections of Markov processes need not be Markov processes again. However here the situation is different. Since the motion time is by construction t and since of the branching time only the increments are important these measures again form Markov processes (see the proof of Theorem 1 for details).

In order to formulate our first theorem we will also need the right continuous inverse A^{-1} of A

$$A^{-1}(\xi,T) := \inf\big\{t>0:\ A(\xi,(0,t])>T\big\} \in [0,\infty]. \tag{1.25}$$

We also use the abbreviation $\Delta(T) = A(A^{-1}(T)) - T \in [0,\infty]$ and agree that $(\Delta(T) + u^{-1})^{-1} = 0$ if either $\Delta(T)=\infty$ or $u=0$. Note that

$$\Delta(T(\gamma)) = \zeta^\Delta_\gamma. \tag{1.26}$$

Theorem 1 ***(i)*** *X and X^h are multiplicative Markov processes. Their log-Laplace transforms are non-negative solutions of*

$$u(s,t,\varphi;x) = \mathbf{E}_{s,x}\left[\varphi(\xi_t) - \int_{A(\xi,s)}^{A(\xi,t)} dT\ \big(\Delta(T) + u^{-1}(A^{-1}(T),t,\varphi;\xi_{A^{-1}(T)})\big)^{-2}\right] \tag{1.27}$$

and $u_h(s,t,\varphi) = h\,u(s,t,\varphi/h)$, where $\varphi\in C_b^+(E)$.

(ii) *If A is continuous, then (1.27) is equivalent to (1.4). In this case, X and X^h are just the ordinary catalytic super process and the catalytic branching particle system, respectively.*

Remark 1.1 **(i)** At this point we were not able to show uniqueness of the solutions of (1.27).

(ii) Note that in the case where A is continuous the term $\Delta(T)$ vanishes. Hence in this case the integral in (1.27) equals

$$\int_{A(\xi,s)}^{A(\xi,t)} dT\, u^2(A^{-1}(T),\varphi;\xi_{A^{-1}(T)}). \tag{1.28}$$

Thus the substitution formula for integrals yields the equivalence of (1.27) to (1.4).

(iii) If A has discontinuities then branching happens while particles stand still. In this case the local number of particles forms an ordinary Galton Watson process (or Feller's branching diffusion, respectively). For this the backward equation $\frac{du}{dT} = -u^2$ has the explicit solution $u(T) = (T + u^{-1})^{-1}$. This explains the extra term $\Delta(T)$ in (1.27).

Example 1.2 We want to derive from Theorem 1 a formula obtained in [BLGLJ97] for catalytic super processes with subordination. That is, A is independent of ξ and is a subordination process (increasing Lévy process). Of course, for two infinitesimal particles the clocks run independently. Technically, to fit this into our framework, we have to consider a subordinator η that is independent of ξ and consider as the new motion process the bivariate process $\xi' := (\xi, \eta)$ on $\mathbf{E} \times [0, \infty)$. The additive functional A of ξ' is simply the second coordinate $A((s,t]) = \eta(t) - \eta(s)$. Theorem 1 tells us how to construct the corresponding super process X'. We further define X for $t \geq 0$ and $B \subset E$ Borel by

$$X_t(B) = X'_t(B \times [0, \infty)).$$

It is clear (due to the independence of ξ and η) that X is again a multiplicative $\mathcal{M}_f(E)$–valued Markov process whose log-Laplace transforms solve (1.27).

For simplicity let us now assume that η is stable with index $\alpha \in (0,1)$. That is, there exists a constant $c > 0$ such that for all $t \geq s$ and all $\lambda \geq 0$

$$\begin{aligned} -\log \mathbf{E}[\exp(-\lambda(\eta(t) - \eta(s)))] &= (t-s)\lambda^\alpha \cdot \frac{c\Gamma(1-\alpha)}{\alpha} \\ &= (t-s)\lambda^\alpha \cdot c \int_0^\infty (1 - e^{-z}) z^{-1-\alpha}\, dz. \end{aligned}$$

Thus

$$W := \sum_{t: A(t) - A(t-) > 0} \delta_{(t, A(t) - A(t-))}$$

is a Poisson point process on $[0,\infty) \times (0, \infty)$ with intensity $cz^{-1-\alpha} dt dz$ and

$$A([s,t)) = \eta(t) - \eta(s) = \int_{[s,t] \times (0,\infty)} z\, W(dr, dz).$$

Hence, abbreviating $u_r = u(r,t,\varphi;\xi_r)$ we get

$$\begin{aligned}
&\mathbf{E}_{s,x}\int_{A(s)}^{A(t)} \big(\Delta(T)+u^{-1}(A^{-1}(T),t,\varphi;\xi_{A^{-1}(T)})\big)^{-2} \\
&\qquad = \mathbf{E}_{s,x}\int_{[s,t]\times(0,\infty)} W(dr,dz)\int_0^z dR\,(R+u_r^{-1})^{-2} \\
&\qquad = \mathbf{E}_{s,x}\int_{[s,t]\times(0,\infty)} W(dr,dz)\big(u_r-(z+u_r^{-1})^{-1}\big) \qquad (1.29)\\
&\qquad = \mathbf{E}_{s,x}\int_s^t dr\int_0^\infty dz\, cz^{-1-\alpha}\frac{z\,u_r^2}{z\,u_r+1} \\
&\qquad = \frac{c\,\pi}{\sin(\pi\alpha)}\,\mathbf{E}_{s,x}\int_s^t dr\,u_r^{1+\alpha}.
\end{aligned}$$

Concluding we get that u is the solution of

$$u(s,t,\varphi;x) = \mathbf{E}_{s,x}\left[\varphi(\xi_t) - \int_0^\infty dr\,\psi(u(r,t,\varphi;\xi_r))\right], \qquad (1.30)$$

where

$$\psi(u) = \frac{c\,\pi}{\sin(\pi\alpha)}u^{1+\alpha}.$$

It is well known that (1.30) has a unique solution. Thus X is the super process with branching rate 1 and (infinite variance) branching law defined by ψ. This result was first obtained in [BLGLJ97, Theorem 8].

1.4 The Branching Time Scale

In some situations it is convenient to have a process with constant branching rate. For example, genealogical considerations are easier in this case. Similarly as in the last section we perform a time change to obtain such a process with constant branching and then do the inverse change to get back the original process.

The Time Change

Now we change from the motion's generic time scale to the generic time scale of the branching. Let $(V_T)_{T\geq 0}$ be the process defined by

$$V_T = ((\xi_t)_{t\leq A^{-1}(T)}, A^{-1}(T), \Delta(T)), \qquad (1.31)$$

where $\Delta(T) = A(A^{-1}(T)) = \inf\{S>0 : A^{-1}(S) > A^{-1}(T)\}$. To avoid technical complications we assume that almost surely $A(t)\uparrow\infty$ as $t\to\infty$. (Otherwise we could assume that $A(t)$ eventually exceeds some fixed value R almost surely and define V_T for $T\leq R$ only.) Then the process V is a well defined Markov right process with values in

$\mathcal{V} := D(\mathbb{R}^+, E) \times [0,\infty) \times \mathbb{R}^+$. (We will use the convention $v = (v^\xi, v^t, v^\Delta)$ for generic points in $\mathcal{V}$.) Hence we can define the branching particle system Y^h where particles branch at rate $1/h$ and move according to V. We can also define the corresponding super process Y with constant branching rate 1.

Note that here we have assumed only that A is an increasing additive functional. We did not use continuity or other regularity assumptions.

The Inverse Time Change

In order to get back to the generic time scale of the motion process we introduce the following stopping times for V

$$\tau_t := \inf\left\{T > 0 : V_T \notin D(\mathbb{R}^+, E) \times [0,t) \times \mathbb{R}^+\right\}, \qquad t \geq 0. \tag{1.32}$$

Clearly, for this process the exit measures $(Y_{\tau_t})_{t\geq 0}$ and $(Y^h_{\tau_t})_{t\geq 0}$ form Markov processes by Dynkin's special Markov property. Note that Y_{τ_t} and $Y^h_{\tau_t}$ are concentrated on $\{v \in \mathcal{V} : v^\Delta = 0\}$.

The final step is to define

$$X_t(C) = Y_{\tau_t}(\{(\xi,\sigma,\Delta) : \xi_t \in C\}), \qquad t \geq 0. \tag{1.33}$$

and

$$X^h_t(C) = Y^h_{\tau_t}(\{(\xi,\sigma,\Delta) : \xi_t \in C\}), \qquad t \geq 0. \tag{1.34}$$

Theorem 2 ***(i)*** $(X_t)_{t\geq 0}$ *is a multiplicative Markov process with values in* $\mathcal{M}_f(E)$*. Its log-Laplace transforms* $u(s,t,\varphi)$ *solve (1.27).* X *is a version of the process described in Theorem 1. If* A *is continuous, then* X *is the classical catalytic super process.*

(ii) $(X^h_t)_{t\geq 0}$ *is a version of the catalytic branching particle system described in Theorem 1 whose log-Laplace transforms are* $u^h(s,t,\varphi) = h\,u(s,t,\varphi/h)$*.*

1.5 The Brownian Snake Construction

We want to profit from the equivalence of the computer time scale γ and the branching time scale T by making an explicit construction of the catalytic processes using the Brownian snake. With this construction we will get

- a pathwise construction of the processes,
- almost sure convergence of the embedded particle system to the super process.

The key is a construction known as the Brownian snake which goes back to Le Gall (see [LG91, LG93, LG99]). In this construction the path of a reflected Brownian motion serves as the (abstract) coding of (i) a series of branching particle systems and (ii) the limiting super process.

Defining the Brownian Snake

Let us start with recalling the Brownian snake. Let $(B_a)_{a\geq 0}$ be a reflected Brownian motion started at $B_0 = 0$. We say that B has an upcrossing of height h (starting) in (a,t) if $B_a = t$ and if there exists a $b > a$ such that $B_b = B_a + h$ and $B_c > B_a$ for all $c \in (a,b)$.

For $h, T, a > 0$ we write

$$N_a^{h,T} = \#\{\text{upcrossings of height } h \text{ in } (b,T),\, b \leq a\}. \tag{1.35}$$

It is an observation of Neveu and Pitman [NP89a, NP89b] that in a Brownian excursion there is a critical binary branching process encoded. To formulate this coding, let $\alpha = \inf\{a > 0 :\, N_a^{h,0} = 1\}$ be the starting point of the first excursion of B beyond the level h and let $\beta = \inf\{b > \alpha :\, B_b = 0\}$ be the end of this excursion. Then $(N_\beta^{h,T})_{T\geq 0}$ is a critical binary branching process with rate h^{-1} and with one ancestor.

If we denote by L_a^T the local time of B up to time a at level T then by the classical result of Lévy

$$L_a^T = 2 \lim_{h\downarrow 0} h N_a^{h,T} \qquad \text{almost surely.} \tag{1.36}$$

If for $z > 0$ we let

$$\alpha(z) = \inf\{a > 0 :\, L_a^0 > 2z\}, \tag{1.37}$$

then $(N_{\alpha(z)}^{h,T})_{T\geq 0}$ is a critical binary branching process with $N_{\alpha(z)}^{h,0}$ being Poisson mean z. Thus

$$\left(h N_{\alpha(z)}^{h,T}\right)_{T\geq 0} \overset{h\to 0}{\longrightarrow} \left(\frac{1}{2} L_{\alpha(z)}^T\right)_{T\geq 0} \qquad \text{almost surely,} \tag{1.38}$$

and $(\frac{1}{2} L_{\alpha(z)}^T)_{T\geq 0}$ is Feller's continuous state branching diffusion starting at z.

The point of this construction of a branching process is that the genealogy is encoded in B. Every $a \in [0, \alpha(z)]$ is the label of an infinitesimal particle alive only at time B_a. Two particles a and b with $a < b$ have a most recent common ancestor $c \in [a,b]$ defined (almost surely uniquely) by

$$B_c = m_{a,b} := \inf\{B_d :\, d \in [a,b]\}. \tag{1.39}$$

Of course, this definition of c is ambiguous if the infimum is zero. In this case, the particles a and b are in different excursions of B and are not related at all.

In order to construct a spatial (super) branching process one has to assign to every particle a a path $(\xi_s^a)_{s\in[0,B_a]}$ of the underlying motion process. This has to be done in such a way that for two particles a and b the paths ξ^a and ξ^b coincide up to the time $m_{a,b}$ when their most recent common ancestor lived.

The above reasoning leads to the following definition.

Definition 1.3 (Brownian Snake) *The Brownian snake associated with the process V (defined in (1.31)) that starts in v is the Markov process $(B_a, \mathbb{V}_a^v)_{a\geq 0}$ with the properties*

1. *$(B_a)_{a\geq 0}$ is a reflected Brownian motion.*

2. *For every a, given B_a, $(\mathbb{V}^v_a(T))_{T\in[0,B_a]}$ is a stopped path in $D([0,B_a],D(\mathbb{R}^+,E)\times(\mathbb{R}^+)^2)$ with the same distribution as $(V_T)_{T\in[0,B_a]}$ with $V_0=v$.*

3. *For $a<b$, $\mathbb{V}^v_a(T)=\mathbb{V}^v_b(T)$ for all $T\le m_{a,b}$.*

4. *For a,b, given B_a, B_b and $\mathbb{V}^v_a(T)$, $T\le m_{a,b}$, the paths $\mathbb{V}^v_a$ and $\mathbb{V}^v_b$ are independent.*

It is easily established by Kolmogorov's extension theorem that such a process exists. If A^{-1} is Lipschitz continuous and if ξ fulfills minimal regularity assumptions, then there exists a continuous version of $\mathbb{V}^v$ that is strong Markov. In general, if A^{-1} is not continuous, one cannot hope for a continuous version of $\mathbb{V}^v$. In fact, in important examples, the property that $a\mapsto\mathbb{V}^a$ is measurable is not even an event (since it is not measurable in the underlying probability space). We will see in a minute what problems arise in this situation.

Regular Version of the Brownian Snake

Let us now formulate conditions that ensure the existence of a continuous, respectively a measurable, version of the Brownian snake. Recall that d is a complete metric on E, let $\bar{d}$ be the corresponding Skorohod metric on $D(\mathbb{R}^+,E)$ and define

$$\tilde{d}(v,w)=\bar{d}(v^\xi,w^\xi)+|v^t-w^t|+|v^\Delta-w^\Delta|.$$

In [BLGLJ97] a weaker metric is used instead of $\tilde{d}$. This allows them to obtain a continuous version of the snake without additional assumptions. Note however that they assume that V is continuous. Without that assumption their metric is too weak to distinguish paths of V appropriately.

The following condition ensures via Kolmogorov's lemma that there exists a (Hölder-) continuous version of the snake which is also strong Markov (see [LG99, Chapter IV.4]).

Condition (C) *There exist constants C, $p>2$, and $\varepsilon>0$ such that for every $v\in\mathcal{V}$ and every $t\ge0$*

$$\mathbf{E}_v\big[\sup\{\tilde{d}(v,V_r),\ r\in[0,t]\}^p\big]\le Ct^{2+\varepsilon}. \tag{1.40}$$

The continuity of $\mathbb{V}^v$ would be needed to construct exit measures via the Brownian snake and to derive the special Markov property. However we do not stress this point here. For the purpose of a representation of the Y-process in terms of local times and the Brownian snake the following condition (compare [LG99, Chapter IV.1, equation (1)]) will be sufficient.

Condition (D) *For every $\varepsilon>0$,*

$$\begin{aligned}&\lim_{\delta\to0}\sup_{s,x}\mathbf{P}_{s,x}\Big[\sup_{r\in[s,s+\delta]}d(x,\xi_r)>\varepsilon\Big]=0\\&\lim_{\delta\to0}\sup_{s,x}\mathbf{P}_{s,x}\big[\sup\{\varrho:\ A([s,s+\varrho])<\delta\}>\varepsilon\big]=0.\end{aligned} \tag{1.41}$$

This condition implies

$$\lim_{\delta\to 0}\sup_v \mathbf{P}_v\Big[\sup_{T\le\delta}\tilde d(v,V_T)>\varepsilon\Big]=0. \tag{1.42}$$

It is well known (see [LG99, Lemma IV.1.1 and the subsequent discussion]) that (1.42) implies that $(B,\mathbb{V}^v)$ has a measurable (in the time coordinate) version. We will henceforth assume that this version is chosen when (D) is in place.

Representation of the Branching Process

Now we formulate how the branching particle system and the super process associated with V can be constructed from the Brownian snake. Let us agree that $d_a N_a^{h,T}$ means integration with respect to the point measure $N_a^{h,T}$ in the variable a. Similarly we use the notation $d_a L_a^T$.

Theorem 3 ***(i)*** *For every $h>0$ the process defined by*

$$Y_T^h := \int_0^{\alpha(z)} d_a N_a^{h,T}\delta_{\mathbb{V}_a^v(T)} \tag{1.43}$$

is a version of the branching particle system associated with V and with the initial state Y_0^h being $\mathcal{H}(h^{-1}z\delta_v)$.
(ii) *For every $T\ge 0$ the almost sure limit $Y_T=\lim_{h\to 0} hY_T^h$ exists and Y is a version of the super process associated with the motion process V.*
(iii) *Define the random measure $\overline{Y}_T$ on $\mathcal{V}\times(0,\infty)$ by $\overline{Y}_T(C\times[h,\infty))=Y_T^h(C)$. Then, given Y, $\overline{Y}_T$ is a Poisson point process with intensity measure $Y_T(dw)\frac{dh}{h^2}$.*
(iv) *If condition (D) holds, then Y_T can be represented in terms of the local times of B at T as*

$$Y_T=\frac{1}{2}\int_0^{\alpha(z)} d_a L_a^T\delta_{\mathbb{V}_a^v(T)}. \tag{1.44}$$

Remark 1.4 If condition (C) is in place, then one can define the local time

$$L_a^{\tau_t}=2\lim_{h\to 0} h\sum_{b\le a} N_{\{b\}}^{h,\tau_t^b}, \tag{1.45}$$

where $N_{\{b\}}^{h,\tau_t^a}=1$ if B has an upcrossing of height h at (b,τ_t^b) and

$$\tau_t^b=\inf\{T>0:\ \mathbb{V}_R^b\notin D(\mathbb{R}^+,E)\times[0,t)\times\mathbb{R}^+\}.$$

The exit measures Y_{τ_t} and $Y_{\tau_t}^h$ can then be defined analogously as in the theorem but with $N_a^{h,T}$ and L_a^T replaced by N_a^{h,τ_t} and $L_a^{\tau_t}$ respectively. (See [LG99, Theorem IV.4.6] for details for the case Y_{τ_t}.) As pointed out in [LG94, end of Section 3] the special Markov property can then be derived from the strong Markov property of the Brownian snake. (For the case where spatial motion is Brownian motion, this was carried out in detail in [LG95, Section 2.3].)

Note however that the special Markov property holds even without condition (C) by results of [Dyn91]. In fact, for the particle system Y^h this is elementary and for the super process Y one could perform the diffusion limit to obtain the special Markov property.

Example for a Non-measurable Snake

Let us now consider an example that displays the difficulty in the case where condition (D) does not hold. Assume that ξ is constant in the time interval $[0,1]$ and is a Brownian motion in the interval $[1,\infty)$. Further let $A(dt) = \mathbf{1}_{[0,1]\cup[2,3]}(t)dt$. That is, branching takes place with rate one for $t \in [0,1]$ or $t \in [2,3]$ and there is no branching at any other time. Hence $A^{-1}(T) = T$ if $T < 1$ and $A^{-1}(T) = T+1$ if $T \in [1,2)$. Thus we have

$$V_T = \begin{cases} \big((\xi_0)_{t\le T}, T, 0\big), & \text{if } T \in [0,1), \\ \big((\xi_t)_{t\le T+1}, T+1, 0\big), & \text{if } T \in [1,2). \end{cases} \tag{1.46}$$

For $d_aL^1_a \otimes d_bL^1_b$–almost all points a,b we have $\mathbf{P}[m_{a,b} < 1 \,|\, L^1] = 1$. (In fact, only countably many points b are endpoints of excursions above level 1. However $d_bL^1_b$ does not have atoms, thus for $d_bL^1_b$–almost all b and all $\varepsilon > 0$ we have $m_{b-\varepsilon,b} < 1$.) Hence the spatial coordinates $((\mathbb{V}^0_a(1)^\xi)_1$ and $((\mathbb{V}^0_b(1)^\xi)_1$ are independent normal random variables. However since measurability depends on uncountably many values of a, this implies that measurability of the map $\mathrm{supp}(L^1) \to \mathbb{R}$, $a \mapsto \mathbb{V}^0_a(1)$ is not an event and hence we cannot define the integral on the l.h.s. of (1.44).

Note however that there is no problem in defining $h\int d_aN^{h,1}_a\delta_{\mathbb{V}^0_a}$ which is just the empirical measure of independent random variables with spatial coordinates being Brownian paths in $[1,2]$. Thus by the law of large numbers as $h \to 0$ this integral converges to the Wiener measure on $C([1,2])$ times the total mass $L^1_{\alpha(z)}$.

1.6 Outline

In the following three subsections we provide the proofs of the theorems one by one.

2 Proof of Theorem 1

Part (ii) was shown in Remark 1.1, hence it remains to show part (i). This amounts to show that

(a) X is a Markov process,

(b) the log-Laplace transforms of X solve (1.27).

(a) Let $t > s \geq 0$ and $\varphi \in C_b^+(E)$ and set $\tilde\varphi(x,t,\Delta) = \varphi(x)$.

$$\begin{aligned}
&\mathbf{E}\Big[e^{-\langle X_t,\varphi\rangle}\big|X_r,\, r\in[0,s]\Big]\\
&\quad= \mathbf{E}\Big[\mathbf{E}\Big[e^{-\langle Z_{\tau_t},\tilde\varphi\rangle}\big|Z_{\tau_r},\, r\in[0,s]\Big]\,\big|X_r,\, r\in[0,s]\Big]\\
&\quad= \mathbf{E}\Big[e^{-\langle Z_{\tau_s},u_{Z,t}(\tilde\varphi)\rangle}\big|X_r,\, r\in[0,s]\Big]\\
&\quad= \mathbf{E}\left[\exp\left(-\int X_s(dx)\,u_{Z,t}\big(\tilde\varphi;(x,s,0)\big)\right)\big|X_r,\, r\in[0,s]\right]\\
&\quad= \exp\left(-\int X_s(dx)\,u_{Z,t}\big(\tilde\varphi;(x,s,0)\big)\right),\\
&\quad= \exp\big(-\langle X_s, u(s,t,\varphi)\rangle\big).
\end{aligned} \tag{2.1}$$

Thus X is a multiplicative Markov process.

(b) As shown above the log-Laplace transforms of X are

$$u(s,t,\varphi;x) = u_{Z,t}(\tilde\varphi;(x,s,0)). \tag{2.2}$$

Hence

$$u(s,t,\varphi;x) = \mathbf{E}_{(x,s,0)}\left[\tilde\varphi(\zeta_{\tau_t}) - \int_0^{\tau_t} d\gamma\, T'(\gamma) u^2_{Z,t}(\tilde\varphi;\zeta_\gamma)\right]. \tag{2.3}$$

The integral equals (see (1.22))

$$\begin{aligned}
&\int_0^{\tau_t} d\gamma\, T'(\gamma)\left(\zeta^\Delta_\gamma + u^{-1}_{Z,t}\big(\tilde\varphi;(\zeta^\xi_\gamma,\zeta^t_\gamma,0)\big)\right)^{-2}\\
&= \int_0^{\tau_t} d\gamma\, T'(\gamma)\left(\Delta(T(\gamma)) + u^{-1}_{Z,t}\big(\tilde\varphi;(\xi_{t(\gamma)},t(\gamma),0)\big)\right)^{-2}.
\end{aligned} \tag{2.4}$$

Note that $t(\gamma) = A^{-1}(T(\gamma))$ for all γ with $T'(\gamma)\neq 0$. Thus (2.4) can be continued by

$$\begin{aligned}
&\int_0^{\tau_t} d\gamma\, T'(\gamma)\left(\Delta(T(\gamma)) + u^{-1}_{Z,t}\big(\tilde\varphi;(\xi_{A^{-1}(T(\gamma))},A^{-1}(T(\gamma)),0)\big)\right)^{-2}\\
&= \int_0^{\tau_t} d\gamma\, T'(\gamma)\Big(\Delta(T(\gamma)) + u^{-1}\big(A^{-1}(T(\gamma)),t,\varphi;\xi_{A^{-1}(T(\gamma))}\big)\Big)^{-2}.
\end{aligned} \tag{2.5}$$

Performing the substitution γ for T yields that (2.5) equals

$$\int_{A(s)}^{A(t)} dT\Big(\Delta(T) + u^{-1}\big(A^{-1}(T),t,\varphi;\xi_{A^{-1}(T)}\big)\Big)^{-2}, \tag{2.6}$$

which shows that u solves (1.27).

□

3 Proof of Theorem 2

We do the proof only for X since the case X^h is even simpler.

Multiplicativity is immediate from the construction. Thus we have to show

(a) that X is Markov

(b) that
$$u(s,t,\varphi;x) := -\log \mathbf{E}_{s,\delta_x}[\exp(-\langle X_t,\varphi\rangle)], \quad s \le t, x \in E, \tag{3.1}$$
solves (1.27),

(c) that X is a version of the process described in Theorem 1.

The procedure is quite similar as in the case of the universal time scale. We begin with some notation and a preliminary lemma.

For $v = (v^\xi, v^t, v^\Delta)$ define the log-Laplace transforms of the Y process (which is time-homogeneous)
$$u_Y(T,\phi;v) = -\log \mathbf{E}_{\delta_v}[\exp(-\langle Y_T,\phi\rangle)]. \tag{3.2}$$
Then u_Y is the unique non-negative solution of
$$u_Y(T,\phi;v) = \mathbf{E}_v\left[\phi(V_T) - \int_0^T dR\, u_Y^2(T-R,\phi;V_R)\right]. \tag{3.3}$$
We further define the log-Laplace transforms of the exit measures Y_{τ_t} by (compare (1.16)ff)
$$u_{Y,t}(\phi;v) = -\log \mathbf{E}_{\delta_v}\left[\exp(-\langle Y_{\tau_t},\phi\rangle)\right]. \tag{3.4}$$
Then
$$u_{Y,t}(\phi;v) = \mathbf{E}_v\left[\phi(V_{\tau_t}) - \int_0^{\tau_t} dR\, u_{Y,t}^2(\phi;V_R)\right] \tag{3.5}$$
and (recall (1.22))
$$u_{Y,t}(\phi;(v^\xi,v^t,v^\Delta)) = \left(\Delta + u_{Y,t}^{-1}(\phi;(v^\xi,v^t,0))\right)^{-1} \tag{3.6}$$
Let $\widetilde{\phi} \in C_b(E\times\mathbb{R}^+)$. Define $\phi(v) = \widetilde{\phi}(v^\xi_{v^t}, v^t)$ and $M_s\widetilde{\phi} : E \to \mathbb{R}$ by
$$M_s\widetilde{\phi}(x) = \mathbf{E}_{s,x}\left[\widetilde{\phi}(\xi_{A^{-1}(A(s))}, A^{-1}(A(s)))\right]. \tag{3.7}$$
Note that $A^{-1}(A(s)) = \inf\{t > s : A((s,t]) > 0\} < \infty$ by the assumption that $A(t) \uparrow \infty$ almost surely. Thus M_s gauges the particles stopped at the first contact with the catalyst. Also note that $A^{-1}(A(s)) = \tau_s$, $\mathbf{P}_{s,x}$–almost surely. Thus, if $v = (v^\xi, s, 0) \in \mathcal{V}$ denotes an arbitrary point with $v^\xi_s = x$ then by the expectation formula for exit measures (see [Dyn91, (1.50)] or [Dyn93, (I.1.20)])
$$M_s\widetilde{\phi}(x) = \langle Y_{\tau_s},\phi\rangle \qquad \mathbf{P}_{s,\delta_v} - \text{a.s.} \tag{3.8}$$

By the Markov property of ξ and since $\widetilde{\phi}$ depends only on the actual position and not on the whole path integrating (3.8) w.r.t. $X_s(dx)$ yields

$$\langle Y_{\tau_s}, \phi\rangle = \langle X_s, M_s\widetilde{\phi}\rangle. \tag{3.9}$$

Furthermore, for $\varphi \in C_b(E)$ define

$$\varphi_t(v) = \begin{cases} \varphi(v^{\xi}_{v^t}) & \text{if } t \leq v^t, \\ 0 & \text{else.} \end{cases} \tag{3.10}$$

Then by the definition of X

$$\langle Y_{\tau_t}, \varphi_t\rangle = \langle X_t, \varphi\rangle. \tag{3.11}$$

Lemma 3.1 *The function $v \mapsto u_{Y,t}(\varphi_t; v)$ depends only on $(v^{\xi}_{v^t}, v^t, v^{\Delta})$. In particular, there exists a map $(s,x) \mapsto \tilde{u}_{Y,t}(s,\varphi;x)$ such that*

$$\tilde{u}_{Y,t}(v^t, \varphi; v^{\xi}_{v^t}) = u_{Y,t}\big(\varphi_t; (v^{\xi}, v^t, 0)\big). \tag{3.12}$$

Moreover

$$\tilde{u}_{Y,t}(s,\varphi) = M_s \tilde{u}_{Y,t}(\,\cdot\,, \varphi, \,\cdot\,). \tag{3.13}$$

Proof This is immediate from (3.5) and the Markov property. □

Corollary 3.2 *For $t \geq s \geq 0$ with the above notation*

$$\mathbf{E}\left[e^{-\langle X_t,\varphi\rangle}\Big| Y_{\tau_r},\, r\in[0,s]\right] = e^{-\langle X_s, \tilde{u}_{Y,t}(s,\varphi)\rangle}. \tag{3.14}$$

Proof By the previous lemma almost surely

$$\begin{aligned} \langle Y_{\tau_s}, u_{Y,t}(\varphi_t)\rangle &= \langle Y_{\tau_s}, \tilde{u}_{Y,t}(s,\varphi)\rangle \\ &= \langle X_s, M_s\tilde{u}_{Y,t}(\,\cdot\,,\varphi,\,\cdot\,)\rangle \\ &= \langle X_s, \tilde{u}_{Y,t}(s,\varphi)\rangle, \end{aligned}$$

where we used (3.9) in the second equality. Thus using (3.11) and the Markov property of $(Y_{\tau_r})_{r\geq 0}$ the l.h.s. of (3.14) equals

$$\begin{aligned} \mathbf{E}\big[e^{-\langle Y_{\tau_t},\varphi_t\rangle}\big|Y_{\tau_r},\, r\in[0,s]\big] &= \mathbf{E}[e^{-\langle Y_{\tau_t},\varphi_t\rangle}\big|Y_{\tau_s}] \\ &= e^{-\langle Y_{\tau_s}, u_{Y,t}(\varphi_t)\rangle} = e^{-\langle X_s, \tilde{u}_{Y,t}(s,\varphi)\rangle}. \end{aligned}$$

□

Now we come to showing (a) and (b).

(a) We prove the Markov property of X. Let $\varphi \in C_b^+(E)$ and let $t \geq s \geq 0$. Then

$$\begin{aligned}\mathbf{E}\Big[e^{-\langle X_t,\varphi\rangle}\Big| X_r,\, r\in[0,s]\Big] \\ &= \mathbf{E}\Big[\mathbf{E}\Big[e^{-\langle X_t,\varphi\rangle}\Big| Y_{\tau_r},\, r\in[0,s]\Big]\Big| X_r,\, r\in[0,s]\Big] \\ &= e^{-\langle X_s, M_s\tilde{u}_{Y,t}(s,\varphi)\rangle}\end{aligned} \tag{3.15}$$

Thus X is Markov.

(b) From Corollary 3.2 we get (recall (3.6))

$$\begin{aligned} u(s,t,\varphi;x) &= \tilde{u}_{Y,t}(s,\varphi) \\ &= u_{Y,t}(\varphi_t;(x_{\leq s},s,0,0)) \\ &= \mathbf{E}_{s,x}\left[\varphi(\xi_t) - \int_0^{\tau_t} dR\, u^2_{Y,t}(\varphi_t;(\xi_r)_{r\leq A^{-1}(R)},A^{-1}(R),\Delta(R)))\right] \\ &= \mathbf{E}_{s,x}\left[\varphi(\xi_t) - \int_0^{\tau_t} dR\left(\Delta(R) + u^{-1}(A^{-1}(R),t,\varphi;\xi_{A^{-1}(R)})\right)^{-2}\right] \\ &= \mathbf{E}_{s,x}\left[\varphi(\xi_t) - \int_{A(s)}^{A(t)} dR\left(\Delta(R) + u^{-1}(r,t,\varphi;\xi_r)\right)^{-2}\right]. \end{aligned} \tag{3.16}$$

However this is exactly what we wanted to show.

(c) As long as we do not have uniqueness of the solutions of the integral equation (1.27) we have to use other means to show that X and X^h coincide with the processes described in Theorem 1.

For any finite measure μ let $\mathcal{H}(\mu)$ denote the Poisson point process with intensity μ. Further let $\mathcal{L}$ denote the law of a random variable. From (1.19) we get that for $h > 0$

$$\mathcal{L}_{\mathcal{H}(\mu/h)}[Z^h_{\tau_t}] = \mathbf{E}_\mu[\mathcal{H}(h^{-1}Z_{\tau_t})]. \tag{3.17}$$

In other words, for Poisson initial data the particle system's exit measure coincides in law with the Poisson process with intensity given by Z_{τ_t}. Using the law of large numbers we get

$$\mathcal{L}_{\mathcal{H}(\mu/h)}[hZ^h_{\tau_t}] \longrightarrow \mathcal{L}[Z_{\tau_t}], \quad h\to 0. \tag{3.18}$$

Hence for X as in Theorem 1 with initial state ν

$$\mathcal{L}_{\mathcal{H}(\nu/h)}[hX^h_t] \longrightarrow \mathcal{L}_\nu[X_t], \quad h\to 0. \tag{3.19}$$

The same reasoning with Z replaced by Y shows that (3.19) also holds for X from Theorem 2. Thus the transition kernels of these Markov processes coincide if the approximating particle systems coincide.

Note that the construction of the two particle systems involves only a finite number of branching points and paths in between these points. It is thus a piece of elementary combinatorics to show that the particle systems coincide. We omit the tedious details. □

4 Proof of Theorem 3

Part (i) is a direct consequence of the construction and the fact that $(N^{h,T}_{\alpha(z)})_{T\geq 0}$ is a binary branching process.

For the case where V is a diffusion process, a detailed proof of this statement can be found in [LG99, page 1423ff]. Clearly, the proof given there does not depend on the special assumption on V but works for any Markov process V. Thus here we content ourselves by giving an outline of the underlying idea.

For $T \geq 0$ let $C_T = \operatorname{supp}(N^{h,T})$ denote the set of those points $a \geq 0$ where an upcrossing of height h above $B_a = T$ starts. The points $a \in C_T$ are interpreted as abstract labels of particles alive at time T. Recall that for $a \leq b$

$$m_{a,b} = \inf\{B_t,\, t \in [a,b]\}.$$

For $a > b$ we write $m_{a,b} = m_{b,a}$ to have a symmetric notation. For $a \in C_T$ and $S \geq T$ we let

$$D^a_{T,S} = \{b \in C_S,\, b \geq a,\, m_{a,b} = T\}.$$

$D^a_{T,S}$ is interpreted as the set of (labels of) descendants of particle a who are alive at time S.

Also for $a \in C_T$ and $t \in [0,T]$ we define

$$\alpha^a_t = \sup\{b \leq a :\, B_b = t,\, m_{a,b} = t\}.$$

Thus α^a_t is the unique $b \in C_t$ with $a \in D^b_{t,T}$ and is interpreted as the ancestor of a which is alive at time t. By construction, for $a, b \in C_T$, $a \neq b$,

$$\alpha^a_t = \alpha^b_t \quad \Longleftrightarrow \quad t < m_{a,b}.$$

Thus $m_{a,b}$ is the time when the most recent common ancestor of a and b splits into two particles $\alpha^a_{m_{a,b}}$ and $\alpha^b_{m_{a,b}}$. Note that the fact that $\alpha^a_{m_{a,b}} \neq \alpha^b_{m_{a,b}}$ reflects the right continuity of the branching particle system.

As noted by Neveu and Pitman (see [NP89a, NP89b]), though with a slightly different formulation, given $N^{h,T}$, the processes $((D^a_{T,S}-a)_{S\geq T},\, a \in C_T)$ are independent identically distributed. Each $(|D^a_{T,S}|)_{S\geq T}$ is a critical binary branching process with mean lifetime h.

By construction, for $T > 0$ and $a \in C_T$,

$$\mathbb{V}^v_a(t) = \mathbb{V}^v_{\alpha_t}(t), \quad t \in [0,T].$$

Thus the motion of a and all its ancestors has followed a path of the V process. Further, by construction, for $a, b \in C_T$ the paths $\mathbb{V}^v_a$ and $\mathbb{V}^v_b$ coincide until time $m_{a,b}$

$$\mathbb{V}^v_a(t) = \mathbb{V}^v_b(t), \quad t \leq m_{a,b}.$$

The evolutions after time $m_{a,b}$ are independent given $\mathbb{V}^v_a(t)$, $t \leq m_{a,b}$. However, this is exactly what happens in a branching particle system when the most recent common ancestor of two particles dies and places two children at its present location.

Part (ii)

For notational simplicity let us assume without loss of generality that almost surely we can distinguish the particles in Y_T^h by their positions, this is

$$Y_T^h = \sum_{w \in \mathrm{supp}(Y_t^h)} \delta_w. \tag{4.1}$$

If this was not the case, we could always enhance the motion process by, say, an independent Brownian motion to get (4.1). Later one could remove the extra Brownian motion by a projection.

Let $\ell(w) = \ell^{T,h}(w)$ denote the remaining lifetime until extinction of w and all its descendants. This is, if

$$Y_S^h = \sum_{w \in \mathrm{supp}(Y_t^h)} Y_{S-T}^{h,w}, \qquad S \geq T,$$

where $(Y^{h,w},\, w \in \mathcal{V})$ is an independent (also independent of Y_T^h) family of branching particle systems with rate h^{-1} and $Y_0^{h,w} = \delta_w$, then

$$\ell(w) := \inf\{t \geq 0 :\, Y_t^{h,w} = 0\}.$$

In the snake construction, if a is such that $w = \mathbb{V}_a^v(T)$, then $\ell(w)$ can be expressed in terms of the height of the excursion of B starting in a and above level T:

$$\ell(w) = \sup\{B_b,\, b \geq a,\, m_{a,b} = a\} - h - T.$$

Hence in the snake construction for $g \geq h$, the measures Y_T^g and Y_T^h are coupled in such a way that

$$Y_T^g(dw) = Y_T^h(dw)\mathbf{1}_{\{\ell(w) \geq g-h\}}. \tag{4.2}$$

Clearly $\{\ell(w),\, w \in \mathcal{V}\}$ is an iid family and is independent of Y_T^h. Thus for $n \in \mathbb{N}$ and for disjoint sets $I^1, \ldots, I^n \subset [h, \infty)$ the measures

$$Y_T^{h,k} := \int Y_T^h(dw)\mathbf{1}_{\{\ell^{T,h}(w)+h \in I^k\}}\delta_w, \qquad k = 1, \ldots, n,$$

form an independent family given Y_T^h. In particular, if $\mathbf{P}[\ell^{T,h}(w) + h \in I^k] = \frac{1}{n}$, $k = 1, \ldots, n$, then $(Y_T^{h,k},\, k = 1, \ldots, n)$ is an exchangeable family.

Finally, note that the distribution of $\ell(w)$ is well-known and is

$$\mathbf{P}[\ell^{T,h}(w) > x] = \frac{h}{h+x}, \qquad x \geq 0.$$

After these general considerations, let us be specific. We fix $\varepsilon > 0$, $n \in \mathbb{N}$ and let $h = \varepsilon/n$ as well as

$$\begin{aligned} I^1 &= [\varepsilon, \infty) \\ I^k &= \left[\frac{\varepsilon}{k}, \frac{\varepsilon}{k-1}\right), \quad k = 2, \ldots, n. \end{aligned}$$

Hence

$$\mathbf{P}[\ell^{T,\varepsilon/n}(w)+\varepsilon/n\in I^1]=\mathbf{P}[\ell^{T,\varepsilon/n}(w)\geq \frac{n-1}{n}\varepsilon]=\frac{1}{n}.$$

Similarly we get $\mathbf{P}[\ell^{T,\varepsilon/n}(w)+\varepsilon/n\in I^k]=\frac{1}{n}$, $k=1,\dots,n$. We thus have that $(Y_T^{\varepsilon,k},\ k=1,\dots,n)$ is exchangeable. However, from (4.2) we know that

$$Y_T^{\varepsilon,1}=Y_T^{\varepsilon}\quad\text{and}\quad Y_T^{\varepsilon,k}=Y_T^{\varepsilon/k}-Y_T^{\varepsilon/(k-1)}.$$

Thus the definition of $Y_T^{\varepsilon,k}$ is independent of n and $(Y_T^{\varepsilon,k},\ k=1,\dots,n)$ extends to an exchangeable family $(Y_T^{\varepsilon,k},\ k\in\mathbb{N})$.

Hence, choosing $\varepsilon=1$, there exists the almost sure limit

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}Y_T^{1,k}=\lim_{n\to\infty}\frac{1}{n}Y_T^{1/n}. \tag{4.3}$$

By definition $h\mapsto Y_T^h$ is monotone decreasing, hence (4.3) implies that

$$Y_T:=\lim_{h\downarrow 0}hY_T^h$$

is well-defined.

Part (iii) Let $C\subset\mathcal{V}$ be measurable and $d>c>0$. By the discussion of Part (ii), for $h\in(0,c)$, given $Y_T^h(C)$, the distribution of $\overline{Y}_T(C\times(c,d))$ is binomial with parameters $Y_T^h(C)$ and $\mathbf{P}[\ell^{T,h}(w)\in(c,d)]=h(d^{-1}-c^{-1})$. It is thus a simple exercise to check that given $Y_T(C)=\lim_{h\downarrow 0}hY_T^h(C)$, the distribution of $\overline{Y}_T(C\times(c,d))$ is Poisson with parameter $(d^{-1}-c^{-1})Y_T(C)$. Furthermore, for disjoint $C_1,C_2\subset\mathcal{V}$, given $Y_T^h(C_1)$ and $Y_T^h(C_2)$, the random variables $\overline{Y}_T(C_1)$ and $\overline{Y}_T(C_2)$ are independent (as the $(\ell^{T,h}(w),\ w\in\mathcal{V})$ are independent). Hence, given Y_T, the random measure $\overline{Y}_T(dw,dh)$ is a Poisson point process with intensity measure $Y_T(dw)\frac{dh}{h^2}$.

Part (iv) In order that the integral

$$\widetilde{Y}_T:=\frac{1}{2}\int_0^{\alpha(z)}d_aL_a^T\delta_{\mathbb{V}_a^v(T)} \tag{4.4}$$

exists, it is necessary and sufficient that $a\mapsto\mathbb{V}_a^v(T)$ is measurable. However this is implied by condition (D) (see [LG99, Lemma IV.1.1 and the discussion following the lemma]).

The rest of this proof consists of showing that given $\widetilde{Y}_T$, the random measure Y_T^h is a Poisson point process with intensity measure $h^{-1}\widetilde{Y}_T$. The strong law of large numbers then yields $h^{-1}Y_T^h\xrightarrow{h\to 0}\widetilde{Y}_T$ almost surely, hence $\widetilde{Y}_T=Y_T$ almost surely.

Note that $(B_a)_{a\in[0,\alpha(z)]}$ consists of a finite number of excursions (from 0) that exceed level T. Clearly, for a and b in different such excursions, $m_{a,b}=0$, thus $\mathbb{V}_a^v$ and $\mathbb{V}_b^v$ are then independent given B. Hence we only have to consider one such excursion of B here.

Our aim is to obtain a convenient *construction* of such an excursion in terms of excursions of B above and below level T. To this end we start with recalling the decomposition of a Brownian motion W (started in $W_0 = 0$) according to its excursions from 0. For details and proofs see [RY99, Chapter XII.2].

An excursion $e = (\zeta^e, (e_t)_{t\in[0,\zeta^e]})$ consists of a lifetime $\zeta^e > 0$ and a continuous function $[0,\zeta^e] \to \mathbb{R}$ with $e^{-1}(\{0\}) = \{0,\zeta^e\}$ (where e^{-1} is the inverse function). Let $\mathcal{E}$ denote the space of such excursions which can be made a Polish space in a canonical way. Let $\mathcal{H}$ be a Poisson point process on $[0,\infty) \times \mathcal{E}$ with intensity measure $\lambda \otimes n$, where λ is Lebesgue measure and n is Ito's excursion measure. Define

$$A(t) = \int_{[0,t]\times\mathcal{E}} \mathcal{H}(ds, de)\, \zeta^e,$$

(which is almost surely finite as $n(\{e : \zeta^e > x\}) \sim x^{-1/2}$) and $A^-(t) = \lim_{s\uparrow t} A(t)$. Note that A is strictly increasing as $n(\mathcal{E}) = \infty$. The idea is to glue together the excursions sampled by $\mathcal{H}$ in "chronological order". Thus $A^-(t)$ will be the starting time and $A(t)$ the end time of an excursion e in the process to be constructed. In order to specify that e we introduce the inverse of A

$$L(a) = \inf\{t > 0 : A(t) > a\}.$$

We will consider L also as a measure $L(da)$ on $\mathbb{R}^+$. Note that $A(L(a)) \geq a \geq A^-(L(a))$ for all $a \geq 0$ and

$$a \in \operatorname{supp}(L) \quad \Longleftrightarrow \quad A^-(L(a)) = a.$$

In the opposite case $a \notin \operatorname{supp}(L)$, there exists a unique atom $e^{L(a)}$ of $\mathcal{H}(\{L(a)\} \times \cdot)$. Clearly

$$\zeta^{e^{L(a)}} = A(L(a)) - A^-(L(a)).$$

We define

$$W_a = \begin{cases} e^{L(a)}_{a-A^-(L(a))}, & a \notin \operatorname{supp}(L), \\ 0, & \text{else} \end{cases}$$

Then $(W_a)_{a\geq 0}$ is a Brownian motion and L is its local time at 0 (see [RY99, Proposition XII.2.5]). Denote

$$\begin{aligned} e^- &= \inf\{e_s,\ s \in [0,\zeta^e]\} \\ e^+ &= \sup\{e_s,\ s \in [0,\zeta^e]\}. \end{aligned}$$

and note that $e^-e^+ = 0$. Let

$$\begin{aligned} \mathcal{H}^- &= \int \mathcal{H}(dt, de)\mathbf{1}_{\{e^-<0\}}\, \delta_{(t,e)} \\ \mathcal{H}^+ &= \int \mathcal{H}(dt, de)\mathbf{1}_{\{e^+>0\}}\, \delta_{(t,e)}. \end{aligned}$$

and note that $\mathcal{H} = \mathcal{H}^- + \mathcal{H}^+$ and $\mathcal{H}^-$ and $\mathcal{H}^+$ are independent Poisson point processes.

Further let $a_1 := \inf\{a > 0 : (e^{L(a)})^- \leq -T\}$ and $t_1 := L(a_1)$, that is

$$t_1 = \inf\big\{t > 0 : \mathcal{H}([0,t] \times \{e : e^- \leq -T\}) > 0\big\}.$$

In words, a_1 is the last time W is in 0 before it first descends to $-T$.

Now let us come back to our reflected Brownian motion B. Let

$$\sigma_0 := \inf\{a > 0 :\ B_a = T\}$$
$$\sigma_2 := \inf\{a > \sigma_0 :\ B_a = 0\}$$
$$\sigma_1 := \sup\{a < \sigma_2 :\ B_a = T\}.$$

Clearly

$$\mathcal{L}\big[(\mathbb{V}^v_a :\ a \in [0,\sigma_2],\ B_a = T)\big|\, B\big] = \mathcal{L}\big[(\mathbb{V}^v_a :\ a \in [\sigma_0,\sigma_1],\ B_a = T)\big|\sigma_0, \sigma_1, (B_a)_{a\in[\sigma_0,\sigma_1]}\big].$$

Now let us assume that the crucial part of B was constructed from W:

$$B_{a+\sigma_0} = W_a, \qquad a \in [0, \sigma_1 - \sigma_0],$$

and $a_1 = \sigma_1 - \sigma_0$ while $(\sigma_0,\ (B_a)_{a\in[0,\sigma_0]})$ and W are independent. Note that

$$\widetilde{Y}^e_T := \frac{1}{2}\int_0^{\sigma_2} d_a L^T_a \delta_{\mathbb{V}^v_{a+\sigma_0}(T)} = \frac{1}{2}\int_0^{a_1} L(da)\delta_{\mathbb{V}^v_{a+\sigma_0}(T)} = \frac{1}{2}\int_0^{t_1} dt\, \delta_{\mathbb{V}^v_{A(t)+\sigma_0}(T)}.$$

Also note that $(\mathbb{V}^v_{A(t)+\sigma_0})_{t\in[0,t_1]}$ does not depend on the details of the excursions but only on σ_0, t_1 and

$$\big(e^- :\ (t,e) \in \operatorname{supp}(\mathcal{H}) \cap [0,t_1] \times \mathcal{E}\big).$$

In particular,

$$(\mathbb{V}^v_{A(t)+\sigma_0})_{t\in[0,t_1]} \quad \text{and } \mathcal{H}^+ \quad \text{are independent.} \tag{4.5}$$

Observe that (almost surely)

$$\operatorname{supp}(N^{T,h}) \subset \operatorname{supp}(L)$$

and that $\mathbb{V}^v_{A^-(t)+\sigma_0} = \mathbb{V}^v_{A(t)+\sigma_0}$ if $W_a > 0$ for (some) $a \in (A^-(t), A(t))$, that is, if a positive excursion starts in $A^-(t)$. Thus

$$\begin{aligned} Y^{e,h}_T := \int_0^{\sigma_2} d_a N^{T,h}_a \delta_{\mathbb{V}^v_{a+\sigma_0}(T)} &= \int \mathcal{H}(dt,de)\mathbf{1}_{\{t_1>t\}}\mathbf{1}_{\{e^+\ge h\}}\delta_{\mathbb{V}^v_{A(t)+\sigma_0}(T)} \\ &= \int \mathcal{H}^+(dt,de)\mathbf{1}_{\{t_1>t\}}\mathbf{1}_{\{e^+\ge h\}}\delta_{\mathbb{V}^v_{A(t)+\sigma_0}(T)}. \end{aligned}$$

By (4.5), $Y^{e,h}_T$ is a Poisson point process with intensity measure $2\widetilde{Y}^e_T \cdot n(\{e :\ e^+ \ge h\})$, given $\widetilde{Y}^e_T$. However, it is well known that $n(\{e :\ e^+ \ge h\}) = (2h)^{-1}$. So we have shown for a single excursion of B that exceeds level T that $Y^{e,h}_T$ is a Poisson process with intensity measure $h^{-1}\widetilde{Y}^e_T$. Adding the corresponding (independent) point processes for the finitely many excursions of B that exceed level T yields that Y^h_T is a Poisson process with intensity measure $h^{-1}\widetilde{Y}_T$ and we are done. □

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