Symmetric (79,27,9)-designs admitting a faithful action of a Frobenius group of order 39[1]

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Abstract

In this paper we present the classification of symmetric designs with parameters $(79, 27, 9)$ on which a nonabelian group of order 39 acts faithfully. In particular, we show that such a group acts semi-standardly with 7 orbits. Using the method of tactical decompositions, we are able to construct exactly 1463 non-isomorphic designs. The orders of the full automorphism groups of these designs all divide $8 \cdot 3 \cdot 13$.

1 Introduction and Basic Notions

Investigations of symmetric block designs have found increasing interest in the field of combinatorics during the past decade. A few methods for the construction of symmetric block designs are known and all of them have shown to be effective in certain situations. Here, we shall use the method of tactical decompositions, assuming that a certain automorphism group operates on the design we want to construct. This method has been suggested and used by Zvonimir Janko [6]; see also [4], [7], and [9].

We assume that the reader is familiar with the basic facts of design theory. For introductory material see, for instance, [2], [3], and [8]. Briefly, a symmetric design with parameters (79,27,9) is a finite incidence structure consisting of two disjoint sets $\mathcal{P}$ and $\mathcal{B}$, where the elements of $\mathcal{P}$ are called points and the elements of $\mathcal{B}$ are called blocks or lines; furthermore, $|\mathcal{P}| = |\mathcal{B}| = 79$. In addition, every block is incident with precisely 27 points and every 2 points are incident with precisely 9 blocks. In this paper, for the sake of simplicity and without loss of generality, we shall say that a point lies on a block or that a block passes through a point if the point and the block in question are incident.

It is known that symmetric $(79, 27, 9)$-designs exist. Namely, the existence of symmetric designs for all triples $(q^{d+1} - q + 1, q^d, q^{d-1})$ has been proved for $d \geq 2$ and q a prime power greater than 2, $q-1$ the order of a projective plane; see [2].

If g is an automorphism of a symmetric design $\mathcal{D}$ with parameters (v, k, λ), then g fixes an equal number of points and blocks, see [8, Theorem 3.1, p. 78]. We denote these fixed sets by $F_{\mathcal{P}}(g)$ and $F_{\mathcal{B}}(g)$ respectively, and their cardinality simply by $|F(g)|$. We shall make use of the following upper bound [8, Corollary 3.7, p. 82] for the number of fixed points:

$$|F(g)| \leq k + \sqrt{k - \lambda} \tag{1}$$

It is also known that an automorphism group G of a symmetric design has the same number of orbits on the set of points $\mathcal{P}$ as on the set of lines $\mathcal{B}$; see [8, Theorem 3.3, p. 79]. Denote that number by t. If there exists a 1-1 mapping between the orbits of G on $\mathcal{P}$ and the orbits of G on $\mathcal{B}$ such that corresponding orbits have the same lengths, then we call the operation of G on the design $\mathcal{D}$ semi-standard.

2 Method of Construction

Let $\mathcal{D}$ be a symmetric design with parameters (v, k, λ), and let G be a subgroup of the automorphism group $Aut(\mathcal{D})$ of $\mathcal{D}$. Denote the point orbits of G on $\mathcal{P}$ by $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_t$ and the line orbits of G on $\mathcal{B}$ by $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_t$. Put $|\mathcal{P}_r| = \omega_r$ and $|\mathcal{B}_i| = \Omega_i$. Obviously,

$$\sum_{r=1}^{t} \omega_r = v \qquad \text{and} \qquad \sum_{i=1}^{t} \Omega_i = v. \tag{2}$$

Let γ_{ir} be the number of points from $\mathcal{P}_r$ which lie on a line from $\mathcal{B}_i$; clearly, that number does not depend on the particular line chosen. Similarly, let Γ_{js} be the number of lines from $\mathcal{B}_j$ which pass through a point from $\mathcal{P}_s$. Then, obviously,

$$\sum_{r=1}^{t} \gamma_{ir} = k \qquad \text{and} \qquad \sum_{j=1}^{t} \Gamma_{js} = k. \tag{3}$$

By [3, Lemma 5.3.1, p. 221], our partition of the point set $\mathcal{P}$ and of the block set $\mathcal{B}$ forms a tactical decomposition of the design $\mathcal{D}$ in the sense of [3, p. 210]. Thus, the following equations hold:

$$\Omega_i \cdot \gamma_{ir} = \omega_r \cdot \Gamma_{ir}, \tag{4}$$

$$\sum_{r=1}^{t} \gamma_{ir} \Gamma_{jr} = \lambda \Omega_j + \delta_{ij}(k - \lambda), \tag{5}$$

$$\sum_{i=1}^{t} \Gamma_{ir} \gamma_{is} = \lambda \omega_s + \delta_{rs}(k - \lambda). \tag{6}$$

For a proof of these equations, the reader is referred to [3] and [4]. Equation (5) together with (4) yields

$$\sum_{r=1}^{t} \frac{\Omega_j}{\omega_r} \gamma_{ir}\gamma_{jr} = \lambda\Omega_j + \delta_{ij}(k-\lambda). \tag{7}$$

Definition 1 *The $(t \times t)$-matrix (γ_{ir}) is called the orbit structure of the design $\mathcal{D}$.*

The first step in the construction of a design is to find all orbit structures. If the number t is not too large, this can be done without the help of a computer. The second step of the construction is usually called indexing. In fact, for each coefficient γ_{ir} of the orbit matrix one has to specify which γ_{ir} points of the point orbit $\mathcal{P}_r$ lie on the lines of the block orbit $\mathcal{B}_i$. Of course, it is enough to do this for a representative of each block orbit, as the other lines of that orbit can be obtained by producing all G-images of that representative. If possible, we choose the line which represents a block orbit in such a way that it is stabilized by a subgroup of G different from $\langle 1 \rangle$.

3 Action of the Frobenius Group of Order 39

We shall determine the action of the non-abelian group G of order 39 on a symmetric (79,27,9)-design $\mathcal{D}$.

Lemma 1 *Let ρ be an element of G with $o(\rho) = 13$. Then $\langle\rho\rangle$ has precisely one fixed point and precisely one fixed block.*

Proof We know that the number of fixed points is the same as the number of fixed blocks for the action of $\langle\rho\rangle$ on $\mathcal{D}$. Denote this number by f. Clearly, $f \equiv 1\,(mod\,13)$, and formula (1) for the upper bound for the number of fixed points yields $f \in \{1, 14, 27\}$. As $o(\rho) > \lambda$, application of a result of M. Aschbacher [1, Lemma 2.6, p. 274] forces the fixed structure to be a subdesign of $\mathcal{D}$. But there is no symmetric design with $v = 14$ or $v = 27$ and $\lambda = 9$. Hence, f is equal to 1. $\square$

Since there is only one isomorphism class of nonabelian groups of order 39, we may put

$$G = \langle\, \rho,\, \sigma \mid \rho^{13} = 1,\;\; \sigma^3 = 1,\;\; \rho^\sigma = \rho^3 \,\rangle.$$

Our next task is to determine the orbit lengths on the set of points and the set of blocks resulting from the action of G on the symmetric design $\mathcal{D}$. The possible orbit lengths are 1, 3, 13 and 39.

Lemma 2 *There is no orbit of length* 3*.*

Proof If false, then ρ would have at least 3 fixed points or 3 fixed lines which, however, is not possible. $\square$

Theorem 1 *The group G acts semi-standardly on $\mathcal{D}$. There are precisely* 7 *orbits on points and on blocks, one of length* 1*, and the other ones each of length* 13*.*

Proof There are precisely three possibilities for the orbit lengths for the point set and the block set. These may be written as arrays:

$$\mathcal{O}_1 = [1, 39, 39], \quad \mathcal{O}_2 = [1, 13, 13, 13, 39], \quad \mathcal{O}_3 = [1, 13, 13, 13, 13, 13, 13].$$

Of course, we may renumber the orbits, if necessary. As the numbers of components of $\mathcal{O}_i$ and $\mathcal{O}_j$ are different for $i \neq j$, the group G acts semi-standardly on $\mathcal{D}$. The case $\mathcal{O}_1$ does not occur, as then it is impossible to construct the fixed block. If we are in the case $\mathcal{O}_2$, then there is no orbit structure. Namely, while trying to compute any two rows of the orbit structure which correspond to line orbits of lengths 1 and 13, we obtain a contradiction to equation (7). Thus, we are in the case $\mathcal{O}_3$. $\square$

In what follows we assume that $\mathcal{P}_1$ contains the fixed point and $\mathcal{B}_1$ the fixed block of $\mathcal{D}$. Thus, $|\mathcal{P}_i| = |\mathcal{B}_i| = 13$ for $i = 2, \dots, 7$. From the structure of G it follows that G acts faithfully on each line and point orbit of length 13. For $i > 1$ we put

$$\mathcal{P}_i = \{{p^i}_0, \dots, {p^i}_{12}\}.$$

Thus, G acts on these point orbits as a permutation group in a unique way. Hence, for the two generators of G we may put

$$\begin{aligned} \rho &= (0,1,2,3,4,5,6,7,8,9,10,11,12) \quad \text{and} \\ \sigma &= (1,3,9)(2,6,5)(4,12,10)(7,8,11). \end{aligned}$$

We immediately obtain the following.

Corollary 1 *The element σ of order 3 of G fixes precisely 7 points and 7 blocks of $\mathcal{D}$. Each block orbit contains a unique line stabilized by σ.*

The following definition is basic for our construction of designs.

Definition 2 *The set of indices of the points of $\mathcal{P}_r$ which lie on a fixed representative of the block orbit $\mathcal{B}_i$ is called the index set for the position (i,r) of the orbit structure and the representative chosen.*

In what follows, we are going to construct a representative for each block orbit; namely, the line fixed by σ. Clearly, σ acts on the intersection of $\mathcal{P}_r$ with the representative of the block orbit $\mathcal{B}_i$. Therefore, the numbers γ_{ir} are all congruent to 0 or to 1 modulo 3. We are now able to compute the orbit structures for $\mathcal{D}$, and we essentially obtain precisely one such structure:

1	13	13	0	0	0	0
1	4	4	6	6	3	3
1	4	4	3	3	6	6
0	6	3	6	3	6	3
0	6	3	3	6	3	6
0	3	6	6	3	3	6
0	3	6	3	6	6	3

It is easy to see that the automorphism group of this orbit structure is isomorphic to the dihedral group D_8 of order 8. We shall use this fact to eliminate isomorphic designs during the indexing process. Here, an automorphism x of a matrix is a permutation of rows followed by a permutation of columns such that the application of x to the matrix leaves the matrix unchanged. It is clear that the set of all such automorphisms is a group, which we call the automorphism group of that matrix.

4 Indexing of the Representatives for each Block Orbit

It is trivial to index - that is, to find the right index sets for - the unique element of $\mathcal{B}_1$, as this is the fixed line of $\mathcal{D}$ under the action of G; this element corresponds to the first row of the orbit structure. Thus, we consider only the right-lower (6×6)-submatrix of the orbit structure; the first row and the first column of our orbit structure are not relevant to our construction. Denote that submatrix by B and its coefficients by b_{ij}, $1 \leq i, j \leq 6$. Obviously, $b_{ij} \in \{3, 4, 6\}$. We want to find all possibilities for the index sets for all the positions of the matrix B. We obtain these possibilities from the cycles of the permutation representation of σ. Clearly, there are precisely four possibilities for the index set in case $b_{ij} = 3$ or $b_{ij} = 4$ and precisely six possibilities for the index set in case $b_{ij} = 6$. All together, we obtain precisely 14 index sets. We write them down and denote them by the non-negative integers from 0 to 13:

$$
\begin{array}{rclcrcl}
\{1,3,9\} & = & 0, & \qquad & \{1,2,3,5,6,9\} & = & 8, \\
\{2,5,6\} & = & 1, & & \{1,3,4,9,10,12\} & = & 9, \\
\{4,10,12\} & = & 2, & & \{1,3,7,8,9,11\} & = & 10, \\
\{7,8,11\} & = & 3, & & \{2,4,5,6,10,12\} & = & 11, \\
\{0,1,3,9\} & = & 4, & & \{2,5,6,7,8,11\} & = & 12, \\
\{0,2,5,6\} & = & 5, & & \{4,7,8,10,11,12\} & = & 13. \\
\{0,4,10,12\} & = & 6, & & & & \\
\{0,7,8,11\} & = & 7, & & & &
\end{array}
$$

We can easily compute the number of possibilities for the index sets that have to be checked for each block representative corresponding to the matrix B. For each of the first two block representatives we obtain $4^4 \cdot 6^2 = 9216$, and for each of the other four representatives we obtain $4^3 \cdot 6^3 = 13824$ possibilities.

Now, one constructs the possible orbits of length 13 one by one. To do this, one considers the rows of B and replaces the numbers b_{ij} by index sets of appropriate size, using the integer names for these index sets. For example, let us take the first row of B. Making use of the ordering of the index sets, the first possibility for an orbit to check would be

$$L_1 : \qquad 4 \quad 4 \quad 8 \quad 8 \quad 0 \quad 0\,.$$

One applies the group $\langle\rho\rangle$ of order 13 to the index sets occuring in L_1 and checks whether two different $\langle\rho\rangle$-images have the right intersection, by adding up the intersection numbers for the six positions of L_1. In this case, the intersection number should be 8 to be good, since the G-fixed-point is not involved. If the intersection condition is satisfied, then we retain L_1; otherwise we discard L_1. The next possibility to check would be

$$4 \quad 4 \quad 8 \quad 8 \quad 0 \quad 1\,,$$

and the last one for the first row of B would be

$$7 \quad 7 \quad 13 \quad 13 \quad 3 \quad 3\,.$$

In this way, one gets six sets which exhaust all the possibilities for the six block orbits, respectively, and which then have to be checked against each other for the intersection property.

To reduce the number of possibilities and to eliminate isomorphic designs as soon as possible, we make use of the group generated by the mapping

$$\alpha \,:\, x \mapsto 2x \;\; (mod\, 13)$$

Clearly, α induces an automorphism of order 12 of $\langle \rho \rangle$ which commutes with σ. It is well known that such a group produces isomorphic designs [4, Lemma 1.8, p. 54]. The cycle decomposition of α on the 14 index sets is

$$(0,1,2,3)\,(4,5,6,7)\,(8,11,13,10)\,(9,12).$$

Hence, α acts as an element of order 4 on the set of index sets.

We have thus used three means for reducing the output of isomorphic symmetric designs; namely, the automorphism group of the orbit structure, the lexicographical ordering of the index sets to obtain an ordering of the orbit or block types, and an ordering of designs (see example L_1 - for a precise explanation of how one introduces such an ordering the reader is referred to [4]), and the group generated by α.

We may summarize the method in the following way. As we remarked above, for the construction of the designs in question, we need not take into account the first row and the first column of our orbit structure. Thus, we get the designs as (6×6)-matrices (so called small incidence matrices) the coefficients of which are from $\{0, ..., 13\}$ representing index sets. Since α acts transitively on the index sets of cardinality 4, we may take into account only such small incidence matrices, which have 4 at position $(1, 1)$. While constructing one small incidence matrix after another, we always apply to it the elements of the automorphism group of the orbit structure. If the resulting matrix has not 4 at the left upper position, we apply a power of α to it to get 4 at $(1, 1)$. Then, if the matrix - so obtained - had

been constructed earlier, we discard it, if not then the matrix will be retained. For the last procedure the ordering of the set of the small incidence matrices is helpful in making the computations very fast.
This proved to be enough to obtain only pairwise non-isomorphic symmetric designs.

5 Results

The computations, outlined above, have been left to a computer. A suitable program was running approximately two hours on a work station. Our main result is contained in the following.

Theorem 2 *There are exactly* 1463 *pairwise non-isomorphic symmetric designs with parameters* $(79, 27, 9)$ *which are faithfully acted upon by a Frobenius group of order* 39*. The full automorphism groups of these designs are all direct products of* $Frob_{39}$ *with a subgroup of* D_8*. The case* $Z_4 \times Frob_{39}$ *does not occur.*

Proof The computer computations led to 1463 symmetric designs. For each design, we computed the statistics for the cardinalities of the intersections of every triple of pairwise different blocks and found that there are 1463 pairwise different statistics. Thus, all designs obtained are pairwise non-isomorphic. The structures of the full automorphism groups of the designs obtained have been determined using the program GAP [5] and a program written by V. Tonchev [10]. □

Application of V. Tonchev's program [10] to each of the 1463 designs resulted in the following statistics for the full automorphism groups of these designs:

$\|Aut\mathcal{D}\|$	39	78	156	312
Number of designs	411	668	312	72

As an immediate consequence of this table we get the following.

Corollary 2 *Up to isomorphisms there are precisely* 441 *symmetric designs with parameters* $(79, 27, 9)$ *that admit the Frobenius group of order* 39 *as full automorphism group.*

From [9] we obtain the following.

Corollary 3 *Up to isomorphisms there are precisely* 72 *symmetric designs with parameters* $(79, 27, 9)$ *the full automorphism group of which is isomorphic to the direct product of a Frobenius group of order* 39 *with a dihedral group of order* 8*.*

Proof S. Pfaff [9] has shown that there are at least 72 symmetric designs with parameters (79,27,9) on which the above direct product acts faithfully. $\square$

6 Some Examples

It would be too much to list all 1463 designs we have constructed. Nevertheless, we want to present four examples of (79,27,9)-designs that have pairwise different orders of their full automorphism groups. We present these designs as (6×6)-matrices the coefficients of which are index sets; here, the starting point is the matrix B introduced earlier. It is not difficult to produce the complete incidence matrices for these designs.

Example 1 $|Aut\mathcal{D}_1| = 39$; $Aut\mathcal{D}_1 \cong Frob_{39}$;

$$\begin{array}{cccccc}
4 & 4 & 8 & 8 & 0 & 2 \\
4 & 4 & 2 & 2 & 12 & 8 \\
12 & 2 & 8 & 2 & 12 & 2 \\
12 & 2 & 2 & 8 & 0 & 8 \\
2 & 12 & 8 & 2 & 0 & 8 \\
0 & 13 & 0 & 12 & 13 & 0
\end{array}$$

Statistics of intersection of every 3 blocks:

Points	0	1	2	3	4	5	6	7	8	9
Triples	338	4654	18616	37219	14443	3224	507	52	0	26

Example 2 $|Aut\mathcal{D}_2| = 78$; $Aut\mathcal{D}_2 \cong Frob_{39} \times Z_2$;

$$\begin{array}{cccccc}
4 & 4 & 8 & 8 & 0 & 0 \\
4 & 4 & 2 & 2 & 12 & 12 \\
11 & 3 & 8 & 2 & 12 & 0 \\
12 & 2 & 2 & 8 & 1 & 10 \\
3 & 11 & 8 & 2 & 0 & 12 \\
2 & 12 & 2 & 8 & 10 & 1
\end{array}$$

Statistics of intersection of every 3 blocks:

Points	0	1	2	3	4	5	6	7	8	9
Triples	572	4342	17914	38896	13260	3536	442	104	0	13

Example 3 $|Aut\mathcal{D}_3| = 156$; $Aut\mathcal{D}_3 \cong Frob_{39} \times E_4$;

$$\begin{array}{cccccc}
4 & 4 & 8 & 8 & 0 & 0 \\
4 & 4 & 2 & 2 & 12 & 12 \\
11 & 3 & 8 & 2 & 10 & 1 \\
11 & 3 & 2 & 8 & 1 & 10 \\
3 & 11 & 8 & 2 & 1 & 10 \\
3 & 11 & 2 & 8 & 10 & 1
\end{array}$$

Statistics of intersection of every 3 blocks:

Points	0	1	2	3	4	5	6	7	8	9
Triples	312	3016	20904	39832	9516	4524	832	104	0	39

Example 4 $|Aut\mathcal{D}_4| = 312$; $Aut\mathcal{D}_4 \cong Frob_{39} \times D_8$;

$$\begin{array}{cccccc}
4 & 4 & 8 & 8 & 2 & 2 \\
4 & 4 & 2 & 2 & 8 & 8 \\
9 & 3 & 10 & 1 & 10 & 1 \\
9 & 3 & 1 & 10 & 1 & 10 \\
3 & 9 & 10 & 1 & 1 & 10 \\
3 & 9 & 1 & 10 & 10 & 1
\end{array}$$

Statistics of intersection of every 3 blocks:

Points	0	1	2	3	4	5	6	7	8	9
Triples	780	3120	18460	41392	11232	3068	676	312	0	39

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