

Extension of Conformal Maps to the Boundary

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The aim of this note is an elementary proof of the following theorem:

Theorem 1 *Let D be a bounded simply connected domain in the complex plane \mathbb{C} . Then the following statements are equivalent:*

- (i) *Every conformal mapping of D onto the unit disk U extends to a homeomorphism of \bar{D} onto \bar{U} .*
- (ii) *The domain D is a Jordan domain.*
- (iii) *Every boundary point of D is simple.*

Comments

1. A **Jordan domain** is a domain whose boundary is the support of a closed Jordan curve. Therefore the implication “(i) \implies (ii)” is an immediate consequence of the Riemann mapping theorem that gives a conformal mapping of D onto U . Extend it. The inverse mapping of the extension gives a homeomorphism of the unit circle onto the boundary ∂D .
2. The implication “(ii) \implies (i)” is the classical extension theorem conjectured by Osgood and proved by Caratheodory [1] and Osgood [6][7]. According to the Jordan Curve Theorem a Jordan domain is automatically bounded and simply connected. In the proof of the above theorem we make no use of the Jordan Curve Theorem. We prove “(ii) \implies (i)” via the chain “(ii) \implies (iii) \implies (i)”.
3. The implication “(iii) \implies (i)” is the version of the extension theorem given by Rudin [8]. A boundary point b of a domain D is **simple**, if every sequence in D with limit b can be connected by a curve $\gamma: [0, 1] \rightarrow \bar{D}$ such that $\gamma(1) = b$ and $\gamma(t) \in D$ for $0 \leq t < 1$. Equivalently we have: Given $\varepsilon > 0$ and a sequence (z_n) in D such that $\lim z_n = b$, almost all z_n are contained in the same connected component of $U_\varepsilon(b) \cap D$, where

$$U_\varepsilon(b) = \{z \in \mathbb{C} \mid |z - b| < \varepsilon\}.$$

Moreover we have a sufficient condition:

Lemma 1 *A boundary point b of D is simple, if for every $\varepsilon > 0$ there is an open neighborhood V of b such that V is contained in $U_\varepsilon(b)$ and $V \cap D$ is connected.*

Novinger [4] has given an elementary proof of “(iii) \implies (i)”. Here we give a simpler proof. The idea is to consider the inverse conformal mapping $f: U \rightarrow D$. So in the proof well-definition and injectivity interchange their roles.

Step 1

Extend f to a continuous map $g: \bar{U} \rightarrow \bar{D}$. This can be done as in the proof of [8, 14.19] using the following lemma.

Lemma 2 *Let (z_n) and (w_n) be sequences in U such that $\lim z_n = \lim w_n = a \in \partial U$. Let $\lim f(z_n) = b_1 \in \partial D$ and $\lim f(w_n) = b_2 \in \partial D$. Then $b_1 = b_2$.*

For the proof see [8, 14.18 (b)].

Step 2

Prove that g is injective on ∂D . This can be done as in [8, 14.18 (a)], but in view of step 1 we don’t need Fatou’s theorem on radial limits – the following well-known lemma is sufficient.

Lemma 3 *Let the function $g: \bar{U} \rightarrow \mathbb{C}$ be continuous in \bar{U} , analytic in U , and constant on a subarc of ∂D . Then g is constant.*

For a simple proof see [2].

The following simplification of step 2 is due to H. J. Fendrich (oral communication): Let a and b be distinct points of ∂D . The points a and b define two closed subarcs A and B of ∂D . Suppose $g(a) = g(b) = c$. We claim that g is constant on A or on B .

Assume the contrary. Then we find $x \in A$ and $y \in B$ such that $g(x) \neq c$ and $g(y) \neq c$. The image $g(C)$ of the chord $C = [x, y]$ is compact, and its distance ε from the point c is positive. Choose sequences $(z_n) \rightarrow a$ and $(w_n) \rightarrow b$ in U . The sequence $g(z_1), g(w_1), g(z_2), g(w_2), \dots$ has limit c . Since c is a simple boundary point, there is a curve γ_m in $U_\varepsilon(c) \cap D$ connecting $g(z_m)$ and $g(w_m)$ for each large m . The curve $g^{-1} \circ \gamma_m$ connects z_m and w_m , therefore it meets C for large m . But the support of γ_m is contained in $U_\varepsilon(c)$ and does not meet $g(C)$. This contradicts the assumption.

The remainder of the proof of “(iii) \implies (i)” follows by an easy compactness argument as in [8].

Finally we prove “(ii) \implies (iii)”. We need a topological result that appears, with an elementary proof, in [3, Appendix to Chapter IX, (Ap. 3.2), and Exercise 2]:

Theorem 2 (JANISZEWSKI) *Let A be a compact and B , a closed subset of the plane \mathbb{C} , a and b two distinct points of $\mathbb{C} - (A \cup B)$ such that neither A nor B separates a and b . Let $A \cap B$ be connected. Then $A \cup B$ does not separate a and b .*

Remark. A set A **separates** two points a and b , if a and b are contained in distinct connected components of $\mathbb{C} - A$.

Now assume (ii). Take a point $b \in \partial D$ and an arbitrary $\varepsilon > 0$. By the continuity of the boundary curve, b is contained in an open subarc J of $\partial D \cap U_\varepsilon(b)$. Therefore we have an open set V such that $V \subseteq U_\varepsilon(b)$ and $J = \partial D \cap V$. Since J is connected, we may suppose V connected. Let w and z be points of $V \cap D$. Then neither $\mathbb{C} - V$ nor ∂D separates w and z , and $\partial D \cap (\mathbb{C} - V)$ is the complementary arc of J , hence connected. By Janiszewski's theorem the set $(\mathbb{C} - V) \cup \partial D$ does not separate w and z . So w and z are in the same connected component of $V - \partial D$. We conclude that $V \cap D$ is connected. The statement (iii) follows by Lemma 1.

Some applications of the extension theorem are given in [5]. Here is another application:

Corollary 1 *Every bounded convex domain in the plane is a Jordan domain.*

Proof. A bounded convex domain is simply connected, and every boundary point is simple. \diamond

References

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