Theoretical Physics 5
Advanced Quantum Mechanics
Winter Semester 2018/2019
Exercise Sheet 10

lecturer: Prof. Dr. Pedro Schwaller — return until: 2019-01-14
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The exercise sheets can be found online at `http://www.staff.uni-mainz.de/pschwal/index_1819.html`.
To be handed in until Monday 2019-01-14 (12:30) to the red letterbox 42 (foyer of Staudingerweg 7).

1. **Proca equation** **(5 points)**
   A massive 4-vector field $\phi_\mu(x)$ interacting with an external 4-current density $k_\mu(x)$ is described by the following Lagrangian:

   $$\mathcal{L} = -\frac{1}{4}\Big(\partial_\mu\phi_\nu - \partial_\nu\phi_\mu\Big)\Big(\partial^\mu\phi^\nu - \partial^\nu\phi^\mu\Big) + \frac{1}{2}\mu^2\phi_\mu\phi^\mu + k_\mu\phi^\mu\,.$$

   (a) **(3 points)** Derive the field equations (*Proca equation*, suggested by Alexandru Proca (1897-1955) in 1934 as an equation to describe massive spin-1 particles, so-called vector mesons).

   (b) **(2 points)** Show that, in contrast to the electromagnetic field A_μ, the continuity equation for k_μ does not follow from the field equations, i.e., one does not have $\partial_\mu k^\mu = 0$. Which condition would be required for the field ϕ_μ if the continuity equation should be true?

2. **Noether's theorem** **(10 points)**
   Consider a Lagrangian $\mathcal{L}[\phi_n, \partial_\mu\phi_n]$ that depends on a set of fields ϕ_n. Let the Lagrangian further exhibit a continous symmetry depending on a real parameter α. Upon infinitesimal changes of the parameter $\alpha \to \alpha + \delta\alpha$ the fields transform as $\phi_n \to \phi_n + \delta\phi_n$.

   (a) **(6 points)** The condition that this is a symmetry of the Lagrangian implies that it is invariant under such transformations, $\frac{\delta\mathcal{L}}{\delta\alpha} = 0$. Rewrite this condition in terms of the variation of the fields $\frac{\delta\phi_n}{\delta\alpha}$ and show that this leads to the following continuity equation:

   $$\partial_\mu J^\mu = 0\,, \qquad J^\mu = \sum_n \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)}\frac{\delta\phi_n}{\delta\alpha}\,.$$

   The current density J^μ is known as a *Noether current*.

   (b) **(4 points)** The Lagrangian of a complex Klein-Gordon field $\mathcal{L} = (\partial_\mu\phi)^*(\partial^\mu\phi) - m^2\phi^*\phi$ is invariant under transformations of the form $\phi \to e^{-i\alpha}\phi,\ \alpha \in \mathbb{R}$. Determine the corresponding Noether current.

3. **Energy momentum tensor** **(10 points)**

   (a) **(6 points)** The energy momentum tensor of a classical field $\phi(x)$ with the Lagrangian $\mathcal{L}(\phi, \partial_\mu \phi)$ reads

   $$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L} \,.$$

   Show that the conservation $\partial_\mu T^{\mu\nu} = 0$ can be derived from invariace of the action under space-time translations of the form $x^\mu \to x^\mu + \xi^\mu$ using Noether's theorem. Take into account that translational invariance is a symmetry of the action (Verify that!) but not of the Lagrangian, and that you therefore have to modify the Noether current correspondingly.

   (b) **(4 points)** Calculate the energy and the momentum of the Klein-Gordon field

   $$H = P^0 = \int \mathrm{d}^3x \, T^{00} \,, \qquad P^i = \int \mathrm{d}^3x \, T^{0i}$$

   expressed in terms of the field and its derivatives.

4. **Probability** **(5 points)**

   In non-relativistic quantum mechanics, the interpretation of the wave function $\psi(\vec{x}, t)$ as a probability amplitude relies on the fact that the probability density $\rho = |\psi|^2$ and the current density

   $$\vec{j} = \frac{\hbar}{2mi} \left[\psi^* (\vec{\nabla} \psi) - (\vec{\nabla} \psi^*) \psi \right]$$

   obey the continuity equation $\dot{\rho} + \vec{\nabla} \cdot \vec{j} = 0$.

   (a) **(1 point)** Derive the continuity equation with the help of the Schrödinger equation for a non-relativistic free particle of mass m.

   (b) **(2 points)** How does the definition of $\vec{j}$ and the derivation of the continuity equation change when the particle interacts with an external electromagnetic field?

   (c) **(2 points)** Analyse the question whether also the field $\phi(\vec{x}, t)$ in the Klein-Gordon equation $(\Box + m^2)\phi = 0$ allows for an analogous interpretation. Why is it impossible in this case to interpret $|\phi|^2$ as a probability density?