Theoretical Physics 5

Advanced Quantum Mechanics

Winter Semester 2018/2019

Exercise Sheet 11

lecturer: Prof. Dr. Pedro Schwaller — return until: 2019-01-21
assistant: Eric Madge — 30 points

The exercise sheets can be found online at `http://www.staff.uni-mainz.de/pschwal/index_1819.html`.
To be handed in until Monday 2019-01-21 (12:30) to the red letterbox 42 (foyer of Staudingerweg 7).

1. **The complex scalar field** **(20 points)**

   Consider a complex scalar field ϕ with the Lagrangian

   $$\mathcal{L} = (\partial_\mu \phi^\dagger)(\partial^\mu \phi) - m^2 \phi^\dagger \phi + \Omega_0 \,. \tag{1}$$

   In the following you can treat ϕ and $\phi^\dagger$ as independent fields.

   (a) **(2 points)** Find the conjugate fields to ϕ and $\phi^\dagger$. Compute the Hamilton operator as a function of the fields and conjugate fields.

   (b) **(4 points)** We now quantize the fields and expand ϕ as

   $$\phi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}}\, e^{-i\,p\cdot x} + b^\dagger_{\vec{p}}\, e^{i\,p\cdot x} \right) ,$$

   where $p \cdot x = p^\mu x_\mu$, $p^\mu = (E_{\vec{p}}, \vec{p})$ and $E_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$. Express $a_{\vec{p}}$ and $b_{\vec{p}}$ in terms of ϕ, $\phi^\dagger$, and their conjugate fields.

   (c) **(4 points)** The field operators obey the canonical equal-time commutation relations

   $$[\phi(t, \vec{x}\,), \pi(t, \vec{y}\,)] = [\phi^\dagger(t, \vec{x}\,), \pi^\dagger(t, \vec{y}\,)] = i\delta^3(\vec{x} - \vec{y}\,) \,,$$

   where all other commutators at same times vanish. Determine the commutators of $a^{(\dagger)}_{\vec{p}}$ and $b^{(\dagger)}_{\vec{p}}$.

   (d) **(4 points)** Diagonalize the Hamilton operator by expressing it in terms of the creation and annihilation operators $a^{(\dagger)}_{\vec{p}}$ and $b^{(\dagger)}_{\vec{p}}$. Which value does Ω_0 have to take to let the vacuum energy vanish?

   (e) **(2 points)** Show that ϕ satisfies the Heisenberg equations of motion.

   (f) **(4 points)** Use the Heisenberg equations of motion to show that the charge

   $$Q = Q_0 + i \int \mathrm{d}^3 x \, \left(\phi^\dagger(x) \pi^\dagger(x) - \pi(x)\phi(x) \right) \tag{2}$$

   is conserved. Write the operator Q as a function of $a_{\vec{p}}$ and $b_{\vec{p}}$ and compute the charge of the states $a^\dagger_{\vec{p}}|0\rangle$ and $b^\dagger_{\vec{p}}|0\rangle$. Which value do you have to assign to Q_0 if the vacuum is not charged?

2. **Two real scalar fields** **(10 points)**

(a) **(1 point)** Express ϕ and $\phi^\dagger$ in the Lagrangian (1) of exercise 1 in terms of two real fields $\varphi_1 = \frac{\phi+\phi^\dagger}{\sqrt{2}}$ and $\varphi_2 = \frac{\phi-\phi^\dagger}{\sqrt{2}i}$, and show that this leads to the following Lagragian:

$$\mathcal{L} = \frac{1}{2}\sum_{j=1}^{2}\left[(\partial_\mu\varphi_j)(\partial^\mu\varphi_j) - m^2\varphi_j^2\right] + \Omega_0\,. \tag{3}$$

(b) **(1 point)** Show that the equations of motions for ϕ and $\phi^\dagger$ from (1) and for φ_1 and φ_2 from (3) are equivalent.

(c) **(1 point)** Now we quantize the real fields:

$$\varphi_j(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left(c_{\vec{p},j}\,e^{-i\,p\cdot x} + c^\dagger_{\vec{p},j}\,e^{i\,p\cdot x}\right)\,, \qquad j = 1,2\,,$$

where $[c_{\vec{p},i}, c^\dagger_{\vec{q},j}] = (2\pi)^3\delta^3(\vec{p}-\vec{q})\delta_{ij}$. Express the ladder operators of the complex field from exercise 1 in terms of the ones of the real fields.

(d) **(3 points)** Rederive the commutator relations of the ladder operators of the complex field from exercise 1 from the commutators of $c^{(\dagger)}_{\vec{p},j}$.

(e) **(2 points)** The Hamilton operator is given in terms of the ladder operators by

$$H = \sum_{j=1}^{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}E_{\vec{p}}\left(c^\dagger_{\vec{p},j}\,c_{\vec{p},j} + \frac{1}{2}[c_{\vec{p},j}, c^\dagger_{\vec{p},j}]\right) - \int \mathrm{d}^3x\,\Omega_0\,.$$

Substitute the ladder operators in terms of the ones of the complex field and show that you obtain the same result as in exercise 1.

(f) **(2 points)** Write the charge operator (2) in terms of $c^{(\dagger)}_{\vec{p},j}$. Are $c^\dagger_{\vec{p},1}|0\rangle$ and $c^\dagger_{\vec{p},2}|0\rangle$ eigenstates of this operator?