Theoretical Physics 5
Advanced Quantum Mechanics
Winter Semester 2018/2019
Exercise Sheet 12

lecturer: Prof. Dr. Pedro Schwaller — return until: 2019-01-28
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The exercise sheets can be found online at `http://www.staff.uni-mainz.de/pschwal/index_1819.html`.
To be handed in until Monday 2019-01-28 (12:30) to the red letterbox 42 (foyer of Staudingerweg 7).

1. **Standard representation of the Dirac matrices** **(3 points)**
   The Dirac matrices γ^μ with $\mu = 0, \ldots, 3$ in standard representation are given by

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},$$

   where σ^i for $i = 1, 2, 3$ are the Pauli matrices. Show that the matrices defined above satisfy the defining equation of the Dirac matrices $\{\gamma^\mu, \gamma^\nu\} = 2\,\eta^{\mu\nu}\,\mathbb{1}_4$.

   Hint: The Pauli matrices satisfy the identity $\sigma^i\sigma^j = i\epsilon^{ijk}\sigma^k + \delta^{ij}\,\mathbb{1}_2$.

2. **Clifford algebra of the Dirac matrices** **(17 points)**
   We consider the Clifford algebra of the Dirac matrices which satisfy $\{\gamma^\mu, \gamma^\nu\} = 2\,\eta^{\mu\nu}\,\mathbb{1}_4$. Furthermore, we define the matrix $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, which satisfies $\{\gamma^5, \gamma^\mu\} = 0$ and $(\gamma^5)^2 = \mathbb{1}_4$.

   (a) **(2 points)** Derive the representation $\gamma^5 = \frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$, where the Levi-Civita symbol is anti-symmetric with respect to index permutations and $\epsilon_{0123} = 1$.

   (b) Show the following trace identities.
      i. **(1 point)** $\mathrm{Tr}(\gamma^\mu) = 0$
      ii. **(3 points)** $\mathrm{Tr}(\gamma^5) = \mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu) = 0$
      iii. **(2 points)** $\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4\eta^{\mu\nu}$
      iv. **(3 points)** $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho})$
      v. **(3 points)** $\mathrm{Tr}(\gamma^{\mu_1}\ldots\gamma^{\mu_{2n+1}}) = \mathrm{Tr}(\gamma^5\gamma^{\mu_1}\ldots\gamma^{\mu_{2n+1}}) = 0$

   (c) **(3 points)** Show that the matrices

$$\Gamma_S = \mathbb{1}_4\,, \quad \Gamma_P = \gamma^5\,, \quad \Gamma_V^\mu = \gamma^\mu\,, \quad \Gamma_A^\mu = \gamma^\mu\gamma^5\,, \quad \Gamma_T^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] = -\Gamma_T^{\nu\mu}$$

   are linearly independent and thereby form a basis of the complex 4×4 matrices. To do so, represent the zero-matrix as a general linear combination

$$A = a\Gamma_S + b\Gamma_P + c_\mu\Gamma_V^\mu + d_\mu\Gamma_A^\mu + e_{\mu\nu}\Gamma_T^{\mu\nu}$$

   with $e_{\mu\nu} = -e_{\nu\mu}$ and deduce that the coefficients vanish by multiplying by suitable Γ matrices and taking the trace.

Hint:
Only use the properties given in the task and no explicit representation. It may help to insert a $\mathbb{1}_4$ within a trace, to rewrite it as a square of a Dirac matrix, and to move a factor within a trace using the anti-commutation relations and the cyclic invariance of the trace.

3. **Feynman slash notation** **(4 points)**
In the lecture we introduced the Feynman slash notation $\not{p} = p_\mu \gamma^\mu$.
   (a) **(2 points)** Show that $\left(\not{p} + m\mathbb{1}_4\right)\left(\not{p} - m\mathbb{1}_4\right) = \left(p^2 - m^2\right)\mathbb{1}_4$.
   (b) **(2 points)** Derive the following identity:

$$\mathrm{Tr}\left[\left(\not{p} - m\mathbb{1}_4\right)\gamma^\mu\left(\not{q} + m\mathbb{1}_4\right)\gamma^\nu\right] = 4\left[p^\mu q^\nu + p^\nu q^\mu - \eta^{\mu\nu}\left(p\cdot q + m^2\right)\right] .$$

4. **Spin sums of Dirac spinors** **(6 points)**
The solutions of the Dirac equation are constructed in the standard representation using the spinors

$$u_s(p) = \frac{1}{\sqrt{p^0 + m}}\begin{pmatrix}(p^0 + m)\chi_s \\ (\vec{\sigma}\cdot\vec{p})\chi_s\end{pmatrix}, \qquad v_s(p) = \frac{1}{\sqrt{p^0 + m}}\begin{pmatrix}(\vec{\sigma}\cdot\vec{p})\phi_s \\ (p^0 + m)\phi_s\end{pmatrix}$$

with $s = \pm\frac{1}{2}$. The Weyl spinors χ_s and ϕ_s satisfy the completeness relations

$$\sum_{s=\pm\frac{1}{2}} \chi_s\chi_s^\dagger = \mathbb{1}_2\,, \qquad \sum_{s=\pm\frac{1}{2}} \phi_s\phi_s^\dagger = \mathbb{1}_2\,.$$

Derive the spin sums

$$\sum_{s=\pm\frac{1}{2}} u_s(p)\bar{u}_s(p) = \not{p} + m\mathbb{1}_4\,, \qquad \sum_{s=\pm\frac{1}{2}} v_s(p)\bar{v}_s(p) = \not{p} - m\mathbb{1}_4\,.$$

$\bar{\psi} = \psi^\dagger\gamma^0$ here describes the Dirac conjugate spinor of ψ.

Hint: Use what you learned in exercises 1 and 3.