Theoretical Physics 5
Advanced Quantum Mechanics
Winter Semester 2018/2019
Exercise Sheet 13

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return until: 2019-02-04
30 points

The exercise sheets can be found online at `http://www.staff.uni-mainz.de/pschwal/index_1819.html`.
To be handed in until Monday 2019-02-04 (12:30) to the red letterbox 42 (foyer of Staudingerweg 7).

1. **Decay of a scalar particle** **(20 points)**

Consider the theory of a real scalar particle ρ with mass M coupled to a complex scalar particle ϕ with mass m. In the following, assume that $M > 2m$. The corresponding Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \rho\, \partial^\mu \rho - M^2 \rho^2\right) + \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi - \kappa\, \rho\, \phi^\dagger \phi\,.$$

(a) **(1 point)** Show that the corresponding Hamilton operator can be written as $H = H_0 + H_{\text{int}}$, where $H_0 = H_0^\rho + H_0^\phi$ includes the Hamilton operators of the free fields and $H_{\text{int}} = \kappa \int \mathrm{d}^3x\, \rho\, \phi^\dagger \phi$ describes the interactions of the fields.

We now work in the interaction picture and quantize the fields,

$$\phi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\, E_{\vec{p},\, m}}} \left(a_{\vec{p}}\, e^{-i\, p_\nu x^\nu} + b^\dagger_{\vec{p}}\, e^{i\, p_\nu x^\nu}\right)\,,$$

$$\rho(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\, E_{\vec{p},\, M}}} \left(c_{\vec{p}}\, e^{-i\, p_\nu x^\nu} + c^\dagger_{\vec{p}}\, e^{i\, p_\nu x^\nu}\right)\,,$$

where $E_{\vec{p},\, m_i} = \sqrt{(\vec{p})^2 + m_i^2}$ and $p^\mu = \left(E_{\vec{p},\, m_i}, \vec{p}\right)$ for $i = \rho, \phi$ with $m_\rho = M,\ m_\phi = m$. From the lecture you know that the scattering matrix element from an initial state $|i\rangle$ at $t = -\infty$ to a final state $|f\rangle$ at $t = +\infty$ is given by

$$S_{fi} = \langle f|S|i\rangle = \langle f|T \exp\left(-i \int_{-\infty}^{\infty} \mathrm{d}t\, H_{\text{int}}(t)\right)|i\rangle\,.$$

(b) **(6 points)** Consider the decay $\rho \to \phi^\dagger \phi$. Compute the S matrix element for $|i\rangle = \sqrt{2\, E_{\vec{p}_1,\, M}}\, c^\dagger_{\vec{p}_1} |0\rangle$ and $|f\rangle = \sqrt{2\, E_{\vec{p}_2,\, m}} \sqrt{2\, E_{\vec{p}_3,\, m}}\, a^\dagger_{\vec{p}_2}\, b^\dagger_{\vec{p}_3} |0\rangle$ to leading order in κ.

(c) **(5 points)** The decay width of a particle of mass M into n particles in its rest frame is given by

$$\Gamma = \frac{1}{2M}\int\left(\prod_j \frac{\mathrm{d}^3 p_j}{(2\pi)^3}\frac{1}{2E_{\vec{p}_j,m_j}}\right)(2\pi)^4\delta^4(P-\sum_j p_j)\,|\mathcal{A}(i\to f)|^2$$

where $\mathcal{A}(i\to f) = \langle f|T|i\rangle$ with $S = 1 + i\,(2\pi)^4\,\delta^4(P-\sum_j p_j)\,T$. P is the four momentum of the decaying particle and the index j runs over all decay products. Compute the decay width from the matrix element calculated above.

(d) **(6 points)** Let us now add another real scalar particle σ with mass $m_\sigma < M/2$ to our theory, which interacts with ρ via the term $\mathcal{L} \supset -\frac{\kappa'}{2}\,\rho\,\sigma^2$. Calculate the decay width for the decay $\rho \to 2\sigma$. Note that you need to include a symmetry factor $\frac{1}{2}$ in the equation for the decay width for a decay into two identical particles.

We now identify ρ with the short-lived neutral Kaon K_S^0, $\phi^{(\dagger)}$ with the charged pions $\pi^\pm$, and σ with the neutral pion π^0. The corresponding masses are $m_{K^0} = 498\,\mathrm{MeV}$, $m_{\pi^\pm} = 140\,\mathrm{MeV}$, $m_{\pi^0} = 135\,\mathrm{MeV}$. Further assume that $\kappa' = \kappa$.

(e) **(1 point)** Compute the branching ratios

$$\mathrm{Br}_{\pi^+\pi^-} = \frac{\Gamma(K_S^0 \to \pi^+\pi^-)}{\Gamma_{\mathrm{tot}}} \qquad \text{and} \qquad \mathrm{Br}_{2\pi^0} = \frac{\Gamma(K_S^0 \to 2\pi^0)}{\Gamma_{\mathrm{tot}}}$$

with the total decay width $\Gamma_{\mathrm{tot}} = \Gamma(K_S^0 \to \pi^+\pi^-) + \Gamma(K_S^0 \to 2\pi^0)$.

(f) **(1 point)** The life time of a particle is given by $\tau = 1/\Gamma_{\mathrm{tot}}$. Which value do you have to assign to κ (in eV) to obtain the literature value $\tau_{K_S^0} = 90\,\mathrm{ps}$?

2. **Spontaneous symmetry breaking** **(10 points)**

Spontaneous symmetry breaking is an important subject. A simple classical example that demonstrates spontaneous symmetry breaking is described by the Lagrangian for a scalar with a negative mass term:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4\,.$$

(a) **(3 points)** How many constants v can you find for which $\phi(x) = v$ is a solution to the equations of motion? Which solution has the lowest energy density (the ground state)?

(b) **(2 points)** The Lagrangian has a symmetry under $\phi \to -\phi$. Show that this symmetry is not respected by the ground state. We say the vacuum expectation value of ϕ is v, and write $\langle\phi\rangle = v$. In this vacuum, the $\mathbb{Z}_2$ symmetry $\phi \to -\phi$ is spontaneously broken.

(c) **(5 points)** Write $\phi(x) = v + \pi(x)$ and substitute back into the Lagrangian. Show that now $\pi = 0$ is a solution to the equations of motion. How does π transform under the $\mathbb{Z}_2$ symmetry $\phi \to -\phi$? Show that this is a symmetry of π's Lagrangian.