Theoretical Physics 5

Advanced Quantum Mechanics

Winter Semester 2018/2019

Exercise Sheet 1

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return until: 2018-10-22
30 points

The exercise sheets can be found online at `http://www.staff.uni-mainz.de/pschwal/index_1819.html`.
To be handed in until Monday 2018-10-22 (12:00) to the red letterbox 42 (foyer of Staudingerweg 7).

1. **Single-particle Hilbert space** **(5 points)**
   Let $\{|x, i\rangle\}$ be an orthonormal basis of a single-particle Hilbert space V_1. x is a continuous index and i is discrete.
   (a) **(1 point)** Which value does the scalar product of two basis elements take?
   (b) **(1 point)** State the completeness relation with respect to the given basis.
   (c) **(3 points)** Expand the states $|\psi\rangle$ and $|\chi\rangle$ in terms of the basis elements. How do you obtain the coefficients of the expansion? Write the scalar product $\langle\psi|\chi\rangle$ in terms of the coefficients. Which condition must hold for the coefficients in order for $|\psi\rangle$ to be a normalized state?

2. **N-particle Hilbert space** **(8 points)**
   Let us construct a Hilbert space of N identical particles

   $$V_N = V_1 \otimes \cdots \otimes V_1\,.$$

   We define N-particle states that decompose into products of single-particle states as follows:

   $$|\alpha_{(N)}\rangle = |\alpha_1, \ldots, \alpha_N\rangle = |\alpha_1\rangle_1 \otimes \cdots \otimes |\alpha_N\rangle_N\,.$$

   V_N is defined as the space of all linear combinations of such product states.
   (a) **(1 point)** Express the scalar product of two product states $|\psi_{(N)}\rangle$ and $|\chi_{(N)}\rangle$ in terms of single-particle scalar products.
   (b) **(1 point)** Write down the scalar product of two general states $|\psi_{(N)}\rangle$ and $|\chi_{(N)}\rangle$ expressing $|\psi_{(N)}\rangle$ and $|\chi_{(N)}\rangle$ as linear combinations of product states.
   (c) **(1 point)** Construct an orthonormal basis for V_N using the basis of V_1
   (d) **(1 point)** State the completeness relation with respect to the constructed basis.
   (e) **(4 points)** Expand a state $|\psi_{(N)}\rangle$ with respect to the basis. How do you obtain the coefficients of the expansion? Which condition must hold for the coefficients in order for $|\psi_{(N)}\rangle$ to be a normalized state? What do you obtain for the coefficients if $|\psi_{(N)}\rangle$ is a product state?

3. **The permutation group** **(17 points)**

Let S_N be the group of permutations of N objects and $\sigma \in S_N$.

(a) **(2 points)** How many elements does the group S_N contain? How many elements does the subgroup of even permutations ($\text{sign}(\sigma) = +1$) contain? (This subgroup is called the alternating group.)

(b) **(6 points)** The quantum operator $\hat{\sigma}$ associated with the permutation σ acts on a product state of N particles as follows,

$$\hat{\sigma}\left[\psi_1(x_1)\ldots\psi_N(x_N)\right] = \psi_{\sigma(1)}(x_1)\ldots\psi_{\sigma(N)}(x_N)\,.$$

Since product states form a basis of the N-particle Hilbert space, the action of $\hat{\sigma}$ on arbitrary states is defined by linearity. The quantum mechanical symmetrization and anti-symmetrization operators Sym and Asym are defined by

$$\text{Sym} = N_S \sum_{\sigma \in S_N} \hat{\sigma}\,, \qquad \text{Asym} = N_A \sum_{\sigma \in S_N} \text{sign}(\sigma)\hat{\sigma}\,,$$

where $N_S,\ N_A$ are normalization factors. Determine $N_S,\ N_A$ such that Sym and Asym are projection operators, i.e. $\text{Sym}^2 = \text{Sym}$, $\text{Asym}^2 = \text{Asym}$. Show that $\text{Sym}\,\text{Asym} = \text{Asym}\,\text{Sym} = 0$. Does the relation $\text{Sym} + \text{Asym} \stackrel{?}{=} 1$ hold?

(c) **(4 points)** Determine the normalization factor C_S for symmetric N-particle wave functions

$$\psi^{(s)}_{c_1\ldots c_N}(x_1,\ldots,x_N) = C_S \sum_{\sigma \in S_N} \psi_{\sigma(c_1)}(x_1)\ldots\psi_{\sigma(c_N)}(x_N)\,,$$

where the single-particle wave functions $\psi_c(x)$ fulfill the following orthonormalization relation:

$$\int \mathrm{d}x\, \psi_b(x)^* \psi_c(x) = \delta_{bc}\,.$$

We assume that all c_i are different.

(d) **(5 points)** Let the dimension of a single-particle Hilbert space V_1 be $(2s+1)$ (this is for instance the case for localized particles with spin s). Compute the dimension of the following spaces:

i. the Hilbert space of product states $V_N = V_1 \otimes \cdots \otimes V_1$ of N particles;

ii. the Hilbert space of symmetric states (bosons) $\text{Sym}\,V_N$;

iii. the Hilbert space of antisymmetric states (fermions) $\text{Asym}\,V_N$.