Theoretical Physics 5
Advanced Quantum Mechanics
Winter Semester 2018/2019
Exercise Sheet 2

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The exercise sheets can be found online at `http://www.staff.uni-mainz.de/pschwal/index_1819.html`.
To be handed in until Monday 2018-10-29 (12:00) to the red letterbox 42 (foyer of Staudingerweg 7).

1. **Bosonic creation and annihilation operators** **(12 points)**

   (a) **(2 points)** Let
   $$|N_1, N_2, \ldots\rangle \equiv \prod_{k=1}^{\infty} \frac{1}{\sqrt{N_k!}} (a_k^\dagger)^{N_k} |0\rangle$$
   be a base state in the occupation number representation of bosons. The creation and annihilation operators act on these states as
   $$\hat{a}_k^\dagger |N_1, N_2, \ldots, N_k, \ldots\rangle = \sqrt{N_k + 1}\, |N_1, N_2, \ldots, N_k + 1, \ldots\rangle\,,$$
   $$\hat{a}_k |N_1, N_2, \ldots, N_k, \ldots\rangle = \sqrt{N_k}\, |N_1, N_2, \ldots, N_k - 1, \ldots\rangle\,.$$
   Derive the commutation relation $[\hat{a}_k, \hat{a}_l^\dagger] = \delta_{kl}$.

   (b) **(4 points)** Compute the following commutators ($\hat{N}_k = \hat{a}_k^\dagger \hat{a}_k,\ \hat{N} = \sum_k \hat{N}_k$):

   i. $[\hat{N}_k, \hat{a}_l]$ ii. $[\hat{N}_k, \hat{a}_l^\dagger]$ iii. $[\hat{N}, \hat{a}_l]$ iv. $[\hat{N}, \hat{a}_l^\dagger]$

   (c) **(6 points)** Using the same notation as in exercise (a), show that
   $$|\phi\rangle = \exp\left(\sum_{k=1}^{\infty} \phi_k \hat{a}_k^\dagger\right)|0\rangle, \qquad \phi_k \in \mathbb{C}$$
   can be expanded in terms of the basis of the Fock states as follows:
   $$|\phi\rangle = \sum_{N_1, N_2, \ldots} \left(\prod_k \frac{1}{\sqrt{N_k!}} \phi_k^{N_k}\right) |N_1, N_2, \ldots\rangle\,.$$
   Show that
   $$\hat{a}_k |\phi\rangle = \phi_k |\phi\rangle \qquad \forall k\,.$$

2. **Fermionic creation and annihilation operators** **(18 points)**

(a) **(3 points)** Starting from the anti-commutation relations for $\hat{a}_k$, $\hat{a}_k^\dagger$

$$\{\hat{a}_k, \hat{a}_{k'}\} = \{\hat{a}_k^\dagger, \hat{a}_{k'}^\dagger\} = 0\,, \qquad \{\hat{a}_k, \hat{a}_{k'}^\dagger\} = \delta_{k,k'}$$

derive the property

$$\hat{N}_k(\hat{N}_k - 1) = 0$$

for $\hat{N}_k \equiv \hat{a}_k^\dagger \hat{a}_k$. Show that for any state $|\psi\rangle$ of N fermions, the expectation values of the occupation number operator, $n_k = \langle\psi|\hat{a}_k^\dagger \hat{a}_k|\psi\rangle$, obey the inequality $0 \leq n_k \leq 1$.

(b) **(4 points)** Compute the following commutators ($\hat{N}_k = \hat{a}_k^\dagger \hat{a}_k,\ \hat{N} = \sum_k \hat{N}_k$):

i. $[\hat{N}_k, \hat{a}_l]$ ii. $[\hat{N}_k, \hat{a}_l^\dagger]$ iii. $[\hat{N}, \hat{a}_l]$ iv. $[\hat{N}, \hat{a}_l^\dagger]$

(c) **(3 points)** Show that fermionic field operators

$$\{\hat{\phi}(\vec{x}), \hat{\phi}(\vec{y})\} = \{\hat{\phi}^\dagger(\vec{x})), \hat{\phi}^\dagger(\vec{y})\} = 0\,, \qquad \{\hat{\phi}(\vec{x}), \hat{\phi}^\dagger(\vec{y})\} = \delta^{(3)}(\vec{x} - \vec{y})$$

satisfy the following relation:

$$\begin{aligned}&\hat{\phi}^\dagger(\vec{x})\hat{\phi}^\dagger(\vec{y})\hat{\phi}(\vec{y})\hat{\phi}(\vec{x})\hat{\phi}^\dagger(\vec{x}_1)\hat{\phi}^\dagger(\vec{x}_2)\hat{\phi}^\dagger(\vec{x}_3)\hat{\phi}^\dagger(\vec{x}_4)|0\rangle \\ &= f(\vec{x}, \vec{y}; \vec{x}_1 \ldots \vec{x}_4)\hat{\phi}^\dagger(\vec{x}_1)\hat{\phi}^\dagger(\vec{x}_2)\hat{\phi}^\dagger(\vec{x}_3)\hat{\phi}^\dagger(\vec{x}_4)|0\rangle\end{aligned}$$

and determine f.

(d) **(4 points)** Show that the operator

$$\hat{V} = \frac{1}{2}\int \mathrm{d}^3x\, \mathrm{d}^3y\, V(\vec{x} - \vec{y})\hat{\phi}^\dagger(\vec{x})\hat{\phi}^\dagger(\vec{y})\hat{\phi}(\vec{y})\hat{\phi}(\vec{x})$$

in the occupation number representation for fermions acts on the state

$$|\psi\rangle = \int \mathrm{d}^3x_1 \ldots \mathrm{d}^3x_N \psi(\vec{x}_1, \ldots, \vec{x}_N)\hat{\phi}^\dagger(\vec{x}_1) \ldots \hat{\phi}^\dagger(\vec{x}_N)|0\rangle$$

as follows:

$$\hat{V}|\psi\rangle = \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{N} \int \mathrm{d}^3x_1 \ldots \mathrm{d}^3x_N V(\vec{x}_i - \vec{x}_j)\psi(\vec{x}_1, \ldots, \vec{x}_N)\hat{\phi}^\dagger(\vec{x}_1) \ldots \hat{\phi}^\dagger(\vec{x}_N)|0\rangle\,.$$

(e) **(4 points)** Let

$$\hat{H} = \hat{H}_0 + \hat{V} \qquad \hat{H}_0 = \int \mathrm{d}^3x\, \hat{\phi}^\dagger(\vec{x}) \left(-\frac{\hbar^2}{2m}\Delta + U(\vec{x})\right)\hat{\phi}(\vec{x})$$

be the Hamilton operator in occupation number representation, where U is an external potential and V is the pair interaction potential from exercise (d). Show that the Schrödinger equation $i\hbar\partial_t|\psi_t\rangle = \hat{H}|\psi_t\rangle$ is satisfied iff the wave function $\psi(\vec{x}_1, \ldots, \vec{x}_N; t)$ satisfies the Schrödinger equation in position space:

$$i\hbar\partial_t\psi(\vec{x}_1, \ldots, \vec{x}_N; t) = \left[\sum_{j=1}^{N}\left(-\frac{\hbar^2}{2m}\Delta_j + U(\vec{x}_j)\right) + \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{N} V(\vec{x}_i - \vec{x}_j)\right]\psi(\vec{x}_1, \ldots, \vec{x}_N; t)$$