Theoretical Physics 5
Advanced Quantum Mechanics
Winter Semester 2018/2019
Exercise Sheet 4

lecturer: Prof. Dr. Pedro Schwaller — return until: 2018-11-19
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The exercise sheets can be found online at `http://www.staff.uni-mainz.de/pschwal/index_1819.html`.
To be handed in until Monday 2018-11-19 (12:30) to the red letterbox 42 (foyer of Staudingerweg 7).

1. **Bogoliubov transformation** **(15 points)**

Consider a dilute gas of weakly interacting bosons at low temperature in a finite volume V. In the Bogoliubov approximation, the corresponding Hamiltonian is given by

$$\hat{H} = \sum_{\vec{p}\neq\vec{0}} \frac{p^2}{2m} \hat{a}^\dagger_{\vec{p}} \hat{a}_{\vec{p}} + \frac{N^2}{2V} \tilde{V}^{(2)}_0 + \frac{N}{2V} \sum_{\vec{p}\neq\vec{0}} \tilde{V}^{(2)}(\vec{p}) \left(\hat{a}^\dagger_{\vec{p}} \hat{a}^\dagger_{-\vec{p}} + 2\, \hat{a}^\dagger_{\vec{p}} \hat{a}_{\vec{p}} + \hat{a}_{\vec{p}} \hat{a}_{-\vec{p}} \right) , \qquad (1)$$

where $\hat{a}^\dagger_{\vec{p}}$ and $\hat{a}_{\vec{p}}$ are bosonic creation and annihilation operators in momentum space. Let further $\tilde{V}^{(2)}(\vec{p})$ be the two-particle-interaction potential with $\tilde{V}^{(2)}(\vec{p}) = \tilde{V}^{(2)}(-\vec{p})$ and $\tilde{V}^{(2)}_0 = \tilde{V}^{(2)}(\vec{0})$. The Bogoliubov transformation of the ladder operators is given by

$$\begin{aligned} \hat{b}_{\vec{p}} &= u_{\vec{p}}\, \hat{a}_{\vec{p}} + v_{\vec{p}}\, \hat{a}^\dagger_{-\vec{p}} , \\ \hat{b}^\dagger_{\vec{p}} &= u^*_{\vec{p}}\, \hat{a}^\dagger_{\vec{p}} + v^*_{\vec{p}}\, \hat{a}_{-\vec{p}} , \end{aligned} \qquad (2)$$

where $u_{\vec{p}}$ and $v_{\vec{p}}$ are complex numbers with

$$u_{\vec{p}} = u_{-\vec{p}} , \qquad v_{\vec{p}} = v_{-\vec{p}} , \qquad |u_{\vec{p}}|^2 - |v_{\vec{p}}|^2 = 1 .$$

(a) **(3 points)** Calculate the following commutators:

i. $\left[\hat{b}_{\vec{p}}, \hat{b}_{\vec{q}}\right]$ ii. $\left[\hat{b}^\dagger_{\vec{p}}, \hat{b}^\dagger_{\vec{q}}\right]$ iii. $\left[\hat{b}_{\vec{p}}, \hat{b}^\dagger_{\vec{q}}\right]$

(b) **(2 points)** Calculate the inverse transformation of (2), i.e. express $\hat{a}^\dagger_{\vec{p}}$ and $\hat{a}_{\vec{p}}$ in terms of $\hat{b}^\dagger_{\vec{p}}$ and $\hat{b}_{\vec{p}}$.

(c) **(3 points)** In the following assume that the coefficients $u_{\vec{p}}$ and $v_{\vec{p}}$ are real. Express the Hamilton operator $\hat{H}$ in terms of the transformed ladder operators $\hat{b}^\dagger_{\vec{p}}$ and $\hat{b}_{\vec{p}}$. Show that the Bogoliubov transformation diagonalizes the Hamilton operator (i.e. that terms proportional to $\hat{b}^\dagger_{\vec{p}} \hat{b}^\dagger_{-\vec{p}}$ and $\hat{b}_{\vec{p}} \hat{b}_{-\vec{p}}$ vanish) if

$$u_{\vec{p}}\, v_{\vec{p}} \frac{p^2}{2m} - \frac{1}{2} \left(u_{\vec{p}} - v_{\vec{p}} \right)^2 \frac{N}{V} \tilde{V}^{(2)}(\vec{p}) = 0 . \qquad (3)$$

(d) **(2 points)** Show that (3) leads to

$$u_{\vec{p}}^2 \left(u_{\vec{p}}^2 - 1\right) = \frac{\left(\frac{N}{V}\tilde{V}^{(2)}(\vec{p}\,)\right)^2}{4\,E_{\vec{p}}^2} \tag{4}$$

and determine $E_{\vec{p}}^2$.

(e) **(5 points)** Use equation (4) to derive expressions for $u_{\vec{p}}^2$, $v_{\vec{p}}^2$ and $u_{\vec{p}}\,v_{\vec{p}}$, and show that the Hamilton operator takes the following form:

$$\hat{H} = \frac{N^2}{2V}\tilde{V}_0^{(2)} + \frac{1}{2}\sum_{\vec{p}\neq\vec{0}}\left(E_{\vec{p}} - \frac{p^2}{2m} - \frac{N}{V}\tilde{V}^{(2)}(\vec{p}\,)\right) + \sum_{\vec{p}\neq\vec{0}} E_{\vec{p}}\,\hat{b}_{\vec{p}}^\dagger\,\hat{b}_{\vec{p}}\,.$$

Determine the ground state energy.

2. **Photon correlation** **(15 points)**

Consider a two-boson state

$$|\Psi_{(2)}\rangle = \int \mathrm{d}^3x_1\,\mathrm{d}^3x_2\,\psi(\vec{x}_1,\vec{x}_2)\hat{\phi}^\dagger(\vec{x}_1)\hat{\phi}^\dagger(\vec{x}_2)|0\rangle$$

with

$$\psi(\vec{x}_1,\vec{x}_2) = c\,\psi_1(\vec{x}_1)\psi_2(\vec{x}_2)\,, \qquad c\in\mathbb{C}\,, \quad \int \mathrm{d}^3x\,|\psi_i(\vec{x}\,)|^2 = 1\,.$$

(a) **(2 points)** Show that the normalization condition $\langle\Psi_{(2)}|\Psi_{(2)}\rangle = 1$ leads to the following normalization of the function $\psi(\vec{x}_1,\vec{x}_2)$:

$$\psi(\vec{x}_1,\vec{x}_2) = \frac{\psi_1(\vec{x}_1)\psi_2(\vec{x}_2)}{\sqrt{1+|\,(\psi_1,\psi_2)\,|^2}}\,,$$

where $(\psi_i,\psi_j) \equiv \int \mathrm{d}^3x\,\psi_i^*(\vec{x}\,)\psi_j(\vec{x}\,)$.

(b) **(6 points)** Calculate the expectation value $\langle\Psi_{(2)}|\hat{n}(\vec{x}\,)|\Psi_{(2)}\rangle$ of the density operator $\hat{n}(\vec{x}\,) \equiv \hat{\phi}^\dagger(\vec{x}\,)\hat{\phi}(\vec{x}\,)$ in the state $|\Psi_{(2)}\rangle$. Which value do you obtain for the integral $\int \mathrm{d}^3x\,\langle\Psi_{(2)}|\hat{n}(\vec{x}\,)|\Psi_{(2)}\rangle$?

(c) **(3 points)** What is the expectation value of the density operator

i. if the one-particle wave functions are orthogonal (i.e. if $(\psi_1,\psi_2) = 0$)?

ii. for the case of two overlapping normal distributions with distance $2a$?

$$\psi_1(\vec{x}\,) = \frac{1}{\pi^{\frac{3}{4}}}e^{-\frac{1}{2}(\vec{x}-a\vec{e}_x)^2}\,, \qquad \psi_2(\vec{x}\,) = \frac{1}{\pi^{\frac{3}{4}}}e^{-\frac{1}{2}(\vec{x}+a\vec{e}_x)^2} \tag{5}$$

(d) **(4 points)** What are the corresponding wave functions in equation (5) and the expectation value of the number density in one dimension? Plot $\langle\Psi_{(2)}|\hat{n}(x)|\Psi_{(2)}\rangle$ and $|\psi_1(x)|^2 + |\psi_2(x)|^2$ for $a = 1$ and $a = 3$.