Theoretical Physics 5

Advanced Quantum Mechanics

Winter Semester 2018/2019

Exercise Sheet 5

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return until: 2018-11-26
30 points

The exercise sheets can be found online at `http://www.staff.uni-mainz.de/pschwal/index_1819.html`.

To be handed in until Monday 2018-11-26 (12:30) to the red letterbox 42 (foyer of Staudingerweg 7).

1. **Holstein-Primakoff Transformation** **(15 points)**

In the Heisenberg model a ferromagnet is described as a cubic lattice of particles with spin S which have spin-interactions with the nearest neighbors. The corresponding Hamilton operator is given by

$$\hat{H} = -J \sum_{\langle k,l \rangle} \vec{\hat{S}}_k \cdot \vec{\hat{S}}_l$$

where k and l denote lattice positions and the sum runs over nearest neighbor pairs. $\vec{\hat{S}}_l$ is the spin operator of the particle at position l and J is the exchange energy. Using the Holstein-Primakoff transformation,

$$\hat{S}_l^+ = \sqrt{2S - \hat{n}_l}\,\hat{a}_l\,, \qquad \hat{S}_l^- = \hat{a}_l^\dagger \sqrt{2S - \hat{n}_l}\,, \qquad \hat{S}_l^z = S - \hat{n}_l\,, \tag{1}$$

where $\hat{S}_l^\pm = \hat{S}_l^x \pm i\hat{S}_l^y$ and $\hat{n}_l = \hat{a}_l^\dagger \hat{a}_l$, the spin operators can be expressed in terms of bosonic creation and annihilation operators $\hat{a}_l^\dagger$ and $\hat{a}_l$. In the following we use natural units ($\hbar = c = 1$).

(a) **(4 points)** Show that the operators defined in (1) satisfy the spin algebra $[\hat{S}_k^a, \hat{S}_l^b] = i\delta_{kl}\epsilon^{abc}\hat{S}_l^c,\ a, b, c \in \{x, y, z\}$.

(b) **(4 points)** Show that $\hat{H}$ to second order in $\hat{a}_l^\dagger$ and $\hat{a}_l$ can be written as follows:

$$\hat{H} = E_0 + JS \sum_{\langle k,l \rangle} \left[\left(\hat{a}_l^\dagger - \hat{a}_k^\dagger \right) \left(\hat{a}_l - \hat{a}_k \right) \right], \qquad E_0 = -\sum_{\langle k,l \rangle} JS^2\,.$$

(c) **(5 points)** Diagonalize the Hamilton Operator using the Fourier transform of the Bose operators

$$\hat{a}_l^\dagger = \frac{1}{\sqrt{N}} \sum_{\vec{p}} e^{-i\,\vec{p}\cdot\vec{x}_l} \hat{a}_{\vec{p}}^\dagger\,, \qquad \hat{a}_l = \frac{1}{\sqrt{N}} \sum_{\vec{p}} e^{i\,\vec{p}\cdot\vec{x}_l} \hat{a}_{\vec{p}}\,,$$

where N is the number of lattice points. You can use the orthogonality relation

$$\frac{1}{N} \sum_l e^{i(\vec{p}-\vec{q})\cdot\vec{x}_l} = \delta_{\vec{p}\vec{q}}$$

and rewrite the sum over nearest neighbors as follows:

$$\sum_{\langle l,k\rangle} f(\vec{x}_l, \vec{x}_k) = \frac{1}{2}\sum_l \sum_{\vec{\delta}} f(\vec{x}_l, \vec{x}_l + \vec{\delta}\,)\,,$$

where the vectors $\vec{\delta}$ point to the nearest neighbors. We neglect boundary effects.

(d) **(2 points)** The operators $\hat{a}^\dagger_{\vec{p}}$ and $\hat{a}_{\vec{p}}$ create or annihilate spin waves with momentum $\vec{p}$, so called magnons. Determine the dispersion relation of the magnons.

2. **Electromagnetism** **(15 points)**
In Lorentz-Heaviside units and with $c = 1$, Maxwell's equations are given by

$$\vec{\nabla}\cdot\vec{E} = \rho, \qquad \vec{\nabla}\times\vec{B} = \frac{\partial \vec{E}}{\partial t} + \vec{j}, \qquad \vec{\nabla}\cdot\vec{B} = 0, \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

By introducing an antisymmetric Tensor $F^{\mu\nu} = -F^{\nu\mu}$ one can make their Lorentz covariance explicit:

$$F^{i0} = E^i\,, \quad F^{ij} = -\sum_{k=1}^{3} \varepsilon^{ijk} B^k\,.$$

Use the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$, $\mu,\nu \in \{0,1,2,3\}$.

(a) **(5 points)** Show that

$$\partial_\mu F^{\mu\nu} = J^\nu \quad \text{and} \quad \partial_{[\mu} F_{\rho\sigma]} = 0$$

reproduces Maxwell's equations with $J^\nu = (\rho, \vec{j}\,)$, $\partial_\mu = (\frac{\partial}{\partial t}, \vec{\nabla})$ and $_{[\ldots]}$ denoting the total antisymmetrisation of the indices.

(b) **(5 points)** Use the transformation properties of the tensor $F^{\mu\nu}$ to show how the electric field $\vec{E}$ transforms under a boost along the x-axis.

(c) **(5 points)** The electromagnetic four-force f^μ acting on a particle with charge e and mass m is given by

$$f^\mu = \frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = eF^\mu{}_\nu \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\,,$$

where τ is the proper time and $p^\mu = (E, \vec{p}\,)$. Show that this definition reproduces the Lorentz force

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = e\left(\vec{E} + \vec{v}\times\vec{B}\right)\,.$$

What does the zero-component of the equation describe?