Theoretical Physics 5
Advanced Quantum Mechanics
Winter Semester 2018/2019
Exercise Sheet 6

lecturer: Prof. Dr. Pedro Schwaller — return until: 2018-12-03
assistant: Eric Madge — 30 points

The exercise sheets can be found online at `http://www.staff.uni-mainz.de/pschwal/index_1819.html`.
To be handed in until Monday 2018-12-03 (12:30) to the red letterbox 42 (foyer of Staudingerweg 7).

1. **Super conductor** **(15 points)**
   In the Bardeen-Cooper-Schrieffer theory of super conductivity, the ground state of a super conductor consists of Cooper pairs and is given by

$$|\psi_0\rangle = \prod_{\vec{p}} \left(u_{\vec{p}} + v_{\vec{p}}\, \hat{a}^\dagger_{\vec{p},\uparrow} \hat{a}^\dagger_{-\vec{p},\downarrow} \right) |0\rangle \,,$$

   where $\hat{a}^\dagger_{\vec{p},\sigma}$ and $\hat{a}_{\vec{p},\sigma}$ are fermionic ladder operators and $u_{\vec{p}}, v_{\vec{p}}, \in \mathbb{C}$. The operator $\hat{a}^\dagger_{\vec{p},\uparrow} \hat{a}^\dagger_{-\vec{p},\downarrow}$ creates a Cooper pair of two electrons with opposite spin and momentum $\vec{p}$, the operator $\hat{a}_{-\vec{p},\downarrow} \hat{a}_{\vec{p},\uparrow}$ annihilates the cooper pair.

   (a) **(7 points)** Calculate the norm $\langle\psi_0|\psi_0\rangle$ of the ground state and show that it is normalized to 1 if $|u_{\vec{p}}|^2 + |v_{\vec{p}}|^2 = 1$.

   (b) **(8 points)** Calculate $\langle\psi_0|\hat{a}_{-\vec{p},\downarrow} \hat{a}_{\vec{p},\uparrow}|\psi_0\rangle$ and $\langle\psi_0|\hat{a}^\dagger_{\vec{p},\uparrow} \hat{a}^\dagger_{-\vec{p},\downarrow}|\psi_0\rangle$.

2. **Wigner crystal** **(15 points)**
   According to a prediction of Wigner,[1] an electron gas at low temperatures and sufficiently low densities exhibits a phase transition into a crystalline structure (bcc). For a qualitative analysis,[2] consider the energy of a lattice of electrons in a homogeneous background of positive charge. Assume that the potential in which each electron moves can be approximated as the potential of a homogeneous, positively charged sphere of radius $r_0 = r_s a_0$ surrounding the electron. Here, r_0 is the mean particle distance in the Wigner crystal with electron density n, i.e. $\frac{4\pi}{3} r_0^3 = 1/n$, and $a_0 = (me^2)^{-1}$ is the Bohr radius. This leads to a model of independent electrons (Einstein approximation) in an oscillator potential

$$\hat{H} = \frac{\hat{\vec{p}}^2}{2m} + \frac{e^2}{2r_0^3}\hat{\vec{x}}^2 - \frac{3e^2}{2r_0} \,.$$

[1] E. P. Wigner. “On the Interaction of Electrons in Metals”. In: *Phys. Rev.* 46 (11 1934), pp. 1002–1011. DOI: 10.1103/PhysRev.46.1002.

[2] E. P. Wigner. “Effects of the electron interaction on the energy levels of electrons in metals”. In: *Trans. Faraday Soc.* 34 (1938), pp. 678–685. DOI: 10.1039/TF9383400678.

Determine the zero-point energy E_0 of this three-dimensional harmonic oscillator and compare the result you obtain to the one known from literature[3]

$$E_0 = \frac{e^2}{2a_0}\left(-\frac{1.792}{r_s} + \frac{2.638}{r_s^{3/2}}\right) .$$

Determine the mean distance of the electrons by minimizing the zero-point energy.

[3]Rosemary A. Coldwell-Horsfall and Alexei A. Maradudin. "Zero-Point Energy of an Electron Lattice". In: *J. Math. Phys.* 1.5 (1960), pp. 395–404. DOI: `10.1063/1.1703670`.