Theoretical Physics 5

Advanced Quantum Mechanics

Winter Semester 2018/2019

Exercise Sheet 7

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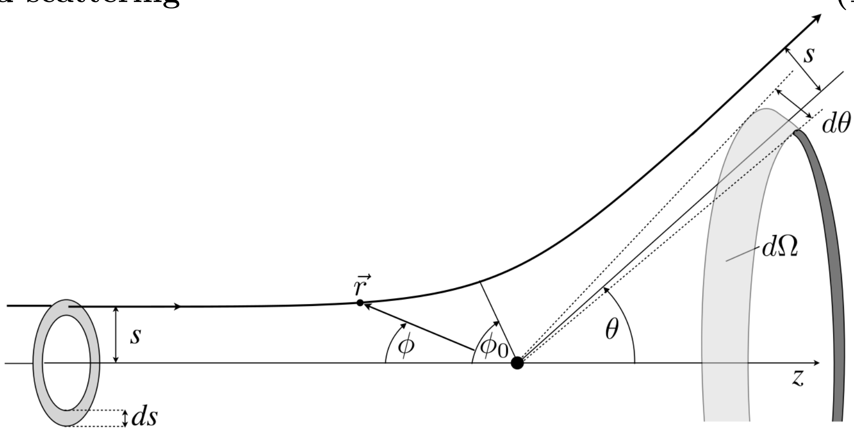
The exercise sheets can be found online at `http://www.staff.uni-mainz.de/pschwal/index_1819.html`.
To be handed in until Monday 2018-12-10 (12:30) to the red letterbox 42 (foyer of Staudingerweg 7).

1. **Rutherford scattering** **(10 points)**



To become familiar with the concept of the differential cross section, we consider a classical scattering process: Rutherford scattering. In this scattering experiment particles of one sort in a homogeneous beam are shot along the z axis at a fixed particle (of an arbitrary other sort) at the coordinate origin. The two particles interact with each other via a repulsive potential of the form

$$V(r) = \frac{\alpha}{r}\,, \qquad \text{where } \alpha > 0 \text{ and } r = |\vec{r}\,|\,,$$

such that an incoming particle with distance s to the z axis is deflected by the scattering angle θ. The interactions satisfies $V(r \to \infty) = 0$. You can thus assume that the incoming particles move freely at large distances before and after scattering. A measure for the distribution after the scattering process is the so called differential cross section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\text{number of scattered particles per time and solid angle}}{\text{number of incoming particles per time and area}}\,.$$

(a) **(3 points)** Taking into account the conservation of particle number, derive a relation between the incoming and outgoing beam and determine the differential cross section. The result is:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{s}{\sin\theta}\left|\frac{\mathrm{d}s}{\mathrm{d}\theta}\right|$$

Justify why we need to take the absolute value of the derivative.

(b) **(7 points)** Use conservation of energy and angular momentum to derive the famous Rutherford cross section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{\alpha}{4E}\right)^2 \sin^{-4}\frac{\theta}{2}\,,$$

where E is the energy of each incoming particle. You can use that we are dealing with a classical process so that each particle moves on a fixed trajectory.

2. **Time evolution operator** **(20 points)**

In the Schrödinger picture quantum mechanical states are time dependent. If a system is in the state $|\psi, t_0\rangle$ at a time t_0, the corresponding state at a time $t > t_0$ is given by (provided that no measurement has been performed in the time interval $[t_0, t]$)

$$|\psi, t\rangle = \hat{U}(t, t_0)|\psi, t_0\rangle\,,$$

where $\hat{U}(t, t_0)$ is the so called time evolution operator.

(a) **(2 points)** Show the following properties of the time evolution operator:

i. $\hat{U}(t_0, t_0) = 1$

ii. $\hat{U}(t, t_0) = \hat{U}(t, t')\hat{U}(t', t_0)$

iii. $\hat{U}^\dagger(t, t_0) = \hat{U}^{-1}(t, t_0)$

iv. $\hat{U}(t, t_0) = \hat{U}^{-1}(t_0, t)$

(b) **(4 points)** Use Schrödinger's equation $i\hbar\frac{\partial}{\partial t}|\psi, t\rangle = \hat{H}(t)|\psi, t\rangle$ with an explicitly time-dependent Hamilton operator and the initial condition (a) i. to derive a differential equation for the time evolution operator and show that this leads to the following integral equation.

$$\hat{U}(t, t_0) = 1 + \frac{1}{i\hbar}\int_{t_0}^{t} \mathrm{d}t'\, \hat{H}(t')\hat{U}(t', t_0) \tag{1}$$

(c) **(5 points)** Show that iterating (1) leads to the Neumann series:

$$\hat{U}(t, t_0) = 1 + \sum_{n=1}^{\infty} \hat{U}^{(n)}(t, t_0)\,,$$

$$\hat{U}^{(n)}(t, t_0) = \left(-\frac{i}{\hbar}\right)^n \underbrace{\int_{t_0}^{t}\mathrm{d}t_1 \int_{t_0}^{t_1}\mathrm{d}t_2 \ldots \int_{t_0}^{t_{n-1}}}_{(t\geq t_1\geq t_2\geq \ldots t_n\geq t_0)} \mathrm{d}t_n\, \hat{H}(t_1)\hat{H}(t_2)\cdots\hat{H}(t_n)\,.$$

Take into account that time dependent Hamilton operators at different times do not necessarily commute.

(d) **(3 points)** We now introduce the Dyson time ordering operator,

$$T\left[\hat{A}(t_1)\hat{B}(t_2)\right] = \begin{cases} \hat{A}(t_1)\hat{B}(t_2) & \text{for } t_1 > t_2\,, \\ \hat{B}(t_2)\hat{A}(t_1) & \text{for } t_2 > t_1\,. \end{cases} \tag{2}$$

Show that

$$\int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2\, \hat{H}(t_1)\hat{H}(t_2) = \frac{1}{2}\int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t} \mathrm{d}t_2\, T\left[\hat{H}(t_1)\hat{H}(t_2)\right]\,.$$

(e) **(5 points)** With the corresponding generalization of (2) to N operators we obtain

$$\int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2 \cdots \int_{t_0}^{t_{N-1}} \mathrm{d}t_N\, \hat{H}(t_1)\cdots\hat{H}(t_N)$$
$$= \frac{1}{N!}\int_{t_0}^{t} \mathrm{d}t_1 \cdots \int_{t_0}^{t} \mathrm{d}t_N\, T\left[\hat{H}(t_1)\cdots\hat{H}(t_N)\right]\,.$$

Show that the time evolution operator can thus be expressed in the following, compact form.

$$\hat{U}(t,t_0) = T\exp\left(-\frac{i}{\hbar}\int_{t_0}^{t} \mathrm{d}t'\, \hat{H}(t')\right)$$

(f) **(1 point)** What is the time evolution operator for the special cases of a time independent hamilton operator and a hamilton operator with $[\hat{H}(t), \hat{H}(t')] = 0$?