Theoretical Physics 5

Advanced Quantum Mechanics

Winter Semester 2018/2019

Exercise Sheet 8

lecturer: Prof. Dr. Pedro Schwaller — return until: 2018-12-17

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The exercise sheets can be found online at `http://www.staff.uni-mainz.de/pschwal/index_1819.html`.
To be handed in until Monday 2018-12-17 (12:30) to the red letterbox 42 (foyer of Staudingerweg 7).

1. **Natural units** **(12 points)**

   In particle physics it is common practice to work in so-called natural units in which the physical constants speed of light c, reduced Planck constant $\hbar$, and Boltzmann constant k_B are set to unity.

   $$c = 1 \qquad \hbar = 1 \qquad k_B = 1$$

   Using this convention all quantities can be expressed in units of powers of energy. According to Einstein, the mass of a particle is for instance related to the corresponding rest energy by the equation $E = mc^2$. For $c = 1$ we obtain $E = m$, i.e. in natural units the mass has the dimension of an energy.[1] As fundamental unit for energy we usually use giga-electronvolts (GeV). The energy dimension of physical quantities and the corresponding conversion factors can be derived unambiguously from the values and SI units of c, $\hbar$, and k_B, or from equations that link the units of various quantities (e.g. $E = mc^2$).

   (a) **(2 points)** What is $1\,\mathrm{GeV}$ in joule? What are c, $\hbar$ and k_B in SI units? To which precision do we know the value of the speed of light? Why?

   (b) **(3 points)** If a quantity Q has energy dimension E^n, we write this as $[Q] = n$. For the mass for instance we obtain $[m] = 1$ since $m \sim E^1$. A dimensionless quantity such as the speed of light has $[c] = 0$ because $c = 1 \sim E^0$. Considering the (in-)equalities

   $$E^2 = m^2c^4 + p^2c^2 \qquad \Delta E\,\Delta t \geq \frac{\hbar}{2} \qquad E_\gamma = \frac{2\pi\hbar c}{\lambda}$$
   $$E_{\text{kin}} = \frac{1}{2}mv^2 \qquad \langle E_{\text{kin}}\rangle = \frac{3}{2}k_BT$$

   derive the energy dimensions of the following quantities:

[1] Of course, we could choose any other quantity as fundamental and for instance express energy in units of mass.

i. time
ii. length
iii. temperature
iv. mass
v. velocity
vi. momentum

(c) **(4 points)** For comparison to experimental data we often need to convert natural units to SI units. What are the SI units of the quantities in exercise (b) i. to vi.? What are their natural units (in GeV)? Derive the corresponding conversion factors (from SI to natural units and vice versa) from the values of c, $\hbar$ and k_B in SI units.

(d) **(3 points)** Consider the position-space Schrödinger equation for an electron in an electromagnetic field.

$$i\hbar\frac{\partial}{\partial t}\psi(\vec{x},t) = \left[\frac{1}{2m_e}\left(-i\hbar\nabla + \frac{e}{c}\vec{A}(\vec{x},t)\right)^2 - e\phi(\vec{x},t)\right]\psi(\vec{x},t)\,.$$

$\vec{A}$ and ϕ are the vector and scalar potentials of the electromagnetic fields and ψ is the (normalized) wave function of the electron.

i. How does this equation read in natural units?
ii. Let the elementary charge e be dimensionless. Derive the energy dimensions of the potentials ϕ and $\vec{A}$.
iii. Which energy dimension does the wave function ψ of the electron have?

2. **Lorentz invariant integral measure** **(6 points)**

(a) **(3 points)** Show that the integral measure

$$\frac{\mathrm{d}^3 p}{2E} \qquad \text{with} \qquad E = \sqrt{\vec{p}^{\,2} + m^2}$$

is invariant under boosts along the z-axis.

(b) **(3 points)** The integral measure $\mathrm{d}^4 p\,\delta(p^2 - m^2)\Theta(p^0)$ is manifestly invariant under proper orthochronous Lorentz transformations (${\Lambda^0}_0 > 0$, $\det\Lambda = 1$). Show that this can be rewritten as the measure from part (a).

3. **The optical theorem** **(12 points)**

The S-matrix describes the transition from an asymptotic (free) initial state $|i\rangle$ at $t \to -\infty$ to an asymptotic (also free) final state $|f\rangle$ at $t \to +\infty$, i.e.

$$|\psi, t = +\infty\rangle = \hat{S}|\psi, t = -\infty\rangle\,.$$

The corresponding matrix elements $\mathcal{A}(i \to f)$ are defined by

$$\langle f|\hat{\mathcal{T}}|i\rangle = (2\pi)^4\delta^4(p_i - p_f)\mathcal{A}(i \to f)\,.$$

Here, the transfer matrix[2] $\mathcal{T}$ is the non-trivial part of the S-matrix, $\hat{S} = 1 + i\hat{\mathcal{T}}$.

(a) **(1 point)** Show that conservation of probability implies the unitarity of the S-matrix and that this implies the following condition for the transfer matrix $\mathcal{T}$:

$$i\left(\hat{\mathcal{T}}^\dagger - \hat{\mathcal{T}}\right) = \hat{\mathcal{T}}^\dagger\hat{\mathcal{T}}$$

[2]Note that $\hat{\mathcal{T}} = (2\pi)^4\delta^4(p_i - p_f)\hat{T}$ compared to the $\hat{T}$ operator defined in the lecture.

(b) **(5 points)** Derive the generalized optical theorem

$$\mathcal{A}(i \to f) - \mathcal{A}^*(f \to i) = i \sum_X \int \mathrm{d}\Pi_X (2\pi)^4 \delta^4(p_i - p_X) \mathcal{A}(i \to X) \mathcal{A}^*(f \to X)$$

using the completeness relation

$$1 = \sum_X \int \mathrm{d}\Pi_X \, |X\rangle\langle X| \,,$$

where the sum is over all N-particle states and we integrate over the corresponding Lorentz invariant phase space, i.e.

$$\mathrm{d}\Pi_X \equiv \prod_{j \in X} \frac{\mathrm{d}^3 p_j}{(2\pi)^3} \frac{1}{2E_j} \,.$$

(c) **(3 points)** Now consider the special case $|i\rangle = |f\rangle = |A\rangle$ with an one-particle state $|A\rangle$ and relate the propagator $\mathcal{A}(A \to A)$ to the total decay width Γ_{tot} of the state $|A\rangle$. The latter is defined by the sum of all partial decay widths into states $|X\rangle$, which in turn are given by

$$\Gamma(A \to X) = \frac{1}{2m_A} \int \mathrm{d}\Pi_X \, (2\pi)^4 \delta^4(p_A - p_X) |\mathcal{A}(A \to X)|^2 \,,$$

where m_A is the mass of the particle.

(d) **(3 points)** Now let $|A\rangle$ ($= |i\rangle = |f\rangle$) be a two-particle state. Derive the corresponding relation between the forward scattering matrix element $\mathcal{A}(A \to A)$ and the total scattering cross section $\sum_X \sigma(A \to X)$, where

$$\sigma(A \to X) = \frac{1}{4E_{\mathrm{CM}} |\vec{p}_{\mathrm{CM}}|} \int \mathrm{d}\Pi_X \, (2\pi)^4 \delta^4(p_A - p_X) |\mathcal{A}(A \to X)|^2 \,.$$

E_{CM} is the center-of-mass energy of the two particles and $|\vec{p}_{\mathrm{CM}}|$ is the magnitude of their momenta in the center-of-mass frame. The relation derived here is often called the "optical theorem".