Theoretical Physics 5
Advanced Quantum Mechanics
Winter Semester 2018/2019
Exercise Sheet 9

lecturer: Prof. Dr. Pedro Schwaller
assistant: Eric Madge

return until: 2019-01-07
40 points

The exercise sheets can be found online at `http://www.staff.uni-mainz.de/pschwal/index_1819.html`.
To be handed in until Monday 2019-01-07 (12:30) to the red letterbox 42 (foyer of Staudingerweg 7).

1. **Basic questions** **(3 points)**
   (a) **(1 point)** What is the Fock space and how is it constructed?
   (b) **(1 point)** Explain the notion of the field operator. Which commutation relations does it fulfil in the fermionic and bosonic case, respectively?
   (c) **(1 point)** Why does the interpretation of the Klein-Gordon equation as a quantum mechanical one-particle theory fail?

In (quantum) field theory we often use the Lagrangian formulation. This has the advantage of making Lorentz invariance manifest. In field theories the Lagrange function L is given by the spatial integral of the Lagrangian density (or just Lagrangian) $\mathcal{L}$, $L = \int \mathrm{d}^3x\,\mathcal{L}$, which in turn is a functional of the fields or field operators and their derivatives.

2. **Euler-Lagrange equations** **(5 points)**
   The action S is defined as the time integral of the Lagrange function,

$$S = \int \mathrm{d}t\, L = \int \mathrm{d}^4x\, \mathcal{L}\,.$$

According to the principle of least action, the fields on which the action depends take a configuration in which the action is stationary. In other words, the action has to be invariant ($\delta S = 0$) under variations of the fields of the form $\phi \to \phi + \delta\phi$ where $\delta\phi$ is an arbitrary field. Consider the case of a Lagrangian that is a functional of a single field ϕ and its derivative only. Show that $\delta S = 0$ leads to the following equation of motion for the field ϕ:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0\,.$$

You can assume that boundary terms vanish.

3. **Some Euler-Lagrange equations** **(14 points)**

Give the field equations corresponding to each of the following Lagrangian densities.

Hint: $\frac{\partial X^\mu}{\partial X^\nu} = \delta^\mu_\nu$ and $X_\mu X^\mu = \eta_{\mu\nu} X^\mu X^\nu$

(a) **(3 points)**

$$\mathcal{L}_{\phi^4} = \frac{1}{2}\Big(\partial_\mu \phi(x)\Big)\Big(\partial^\mu \phi(x)\Big) - \frac{m^2}{2}\,\phi(x)^2 - \frac{\lambda}{4!}\,\phi(x)^4$$

for $\phi(x) \in \mathbb{R}\,,\ m > 0\,,\ \lambda > 0\,.$

(b) **(3 points)**

$$\mathcal{L}_{\phi^3} = \Big(\partial_\mu \phi_a(x)\Big)\Big(\partial^\mu \phi^a(x)\Big) - \phi_a(x) M^{ab} \phi_b(x) - g\Gamma^{abc}\phi_a(x)\phi_b(x)\phi_c(x)$$

for $\phi_a(x) \in \mathbb{R}$, $1 \le a \le N$, M and Γ each real and symmetric.

(c) **(3 points)**

$$\mathcal{L}_{\text{em}} = -\frac{1}{4}\Big(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)\Big)\Big(\partial^\mu A^\nu(x) - \partial^\nu A^\mu(x)\Big)$$

for $A_\mu(x) \in \mathbb{R}\,.$

(d) **(5 points)**

$$\mathcal{L}_{\text{sem}} = \mathcal{L}_{\text{em}} + \Big(\partial_\mu \phi(x) + ieA_\mu(x)\phi(x)\Big)^* \Big(\partial^\mu \phi(x) + ieA^\mu(x)\phi(x)\Big)$$

for $\phi(x) \in \mathbb{C}\,,\ A_\mu(x) \in \mathbb{R}\,,\ e > 0\,.$

4. **Maxwell's equations** **(4 points)**

The field strength tensor $F^{\mu\nu}$ is defined as $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$.

(a) **(2 points)** Show that the equation of motion for A^μ from exercise 3. (d) can be written as $\partial_\mu F^{\mu\nu} = j^\nu$. What is the corresponding definition of the four-current density j^μ?

(b) **(2 points)** Show that the field strength tensor satisfies the Bianchi identity $\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$. Further show that this leads to Maxwell's equation $\partial_{[\mu} F_{\nu\lambda]} = 0$.

5. **Hamiltonian density** **(14 points)**

In field theories, the Hamilton function is defined by the spatial integral of the Hamilton density $\mathcal{H}$, $H = \int \mathrm{d}^3x\,\mathcal{H}$. The latter is a functional of the fields ϕ and their respective conjugate momenta π and is related to the Lagrangian density via a Legendre transform:

$$\mathcal{H}[\phi,\pi] = \pi\,\dot{\phi}[\phi,\pi] - \mathcal{L}[\phi,\,\dot{\phi}[\phi,\pi]]\,, \qquad \pi = \frac{\partial \mathcal{L}[\phi,\dot{\phi}]}{\partial \dot{\phi}}\,.$$

Determine the Hamiltonian densities corresponding to the Lagrangian densities in exercise 3.

Hint: Here, ϕ is a general field and may carry indices (e.g. ϕ_a or A_μ). In this case the corresponding conjugate momentum carries the same indices.