Theoretical Physics 5

- Advanced Quantum Mechanics

Introduce

Date / Time: Monday, Wednesday 10-12 = 10:15 - 11:55 (?)

Exercises : 3 groups, Wednesday 8-10 12-2 2-4

↳ pass around sheet, indicate if no time works

Exam: Written, Feb 19, 2019 9:00

Questions?

Goal of the lecture (→ Modulhandbuch)

Many body theory

Formalism: Fock space, Spin Statistics, 2nd Quantisation

Applications: Superconductivity, Bose Einstein condensate

Relativistic Quantum mechanics

Klein-Gordon eqn, Dirac eqn, Lorentz group
Towards quantum field theory

Path integral formulation of QM

Literature

Weinzierl script QM 2

Schwabl, Advanced QM
Sakurai, Advanced QM (a bit outdated)
Srednicki, QFT
Peskin Schröder, QFT
Ryder, QFT

Reminder: Quantum mechanics

Single particle, one dimension

- $\psi(x,t)$ describes state of system
  $|\psi(x,t)|^2 dx$ probability to find particle at place x at time t
- Observables: Hermitian operators $\hat{O}$
- Expectation value

$$\langle \hat{O} \rangle = (\psi, \hat{O}\psi) = \int_{-\infty}^{+\infty} dx\, \psi^*(x,t)\, \hat{O}\, \psi(x,t)$$

(Schrödinger picture)

- Schrödinger eqn

$$i\hbar \frac{\partial}{\partial t} \psi(x,t) = \hat{H}\, \psi(x,t)$$

Note: Will often set $\hbar = 1$, $c = 1$ (natural units)

- Collapse of wave function

Let $\hat{O}$ be a herm. operator, eigensystem ψ_n, λ_n.
If a measurement gives λ_j, afterwards the system will be in state ψ_j.

Dirac notation

Wave functions are vectors in Hilbert space, e.g. $\psi(x,t)$ is an element of L^2, the space of square integrable functions

$f \in L^2(\mathbb{R}, \mathbb{C}) \Leftrightarrow \int_{-\infty}^{+\infty} dx\, |f(x)|^2 < \infty$.

In a more basis independent way, we write

$$|\psi\rangle \in V$$

for elements of the Hilbert space V. Ket-vector.

~~Examples: $|n\rangle$ $|p\rangle$ $|\vec{p}\rangle$~~

~~discrete quantum number~~ ~~continuous qn's~~

Concrete examples:

Let $\psi_n(x)$ be eigenstates of an operator with discrete quantum number n. Then we write $\psi_n(x) \to |n\rangle$

$$|\psi\rangle = \sum_n c_n |n\rangle$$

Similarly for eigenstates with continuous quantum numbers, e.g. momentum eigenstates $\psi_p(x) \to |p\rangle$

$$|\psi\rangle = \int dp \; c(p)\, |p\rangle$$

Exercise: States with continuous and discrete quantum numbers.
↳ do not confuse with continuous + discrete spectrum!

Dual basis: Bra-vectors $\langle\psi|$

Scalar product: $\langle\psi|\chi\rangle$

Orthogonality:
$$\langle n|m\rangle = \delta_{nm} \quad \text{(discrete)}$$
$$\langle p|p'\rangle = \delta(p-p') \quad \text{(continuous)}$$

Completeness:
$$\sum |n\rangle\langle n| = 1 \quad \leftarrow \text{operator}$$
$$\int dp\, |p\rangle\langle p| = 1$$

Operators: $\langle \psi | \hat{O} | \chi \rangle \equiv \langle \psi | \underbrace{\hat{O}\, \chi \rangle}_{\hat{O}\, \chi(x)}$

Let $\{ |1\rangle, |2\rangle, \dots \}$ be an orthonormal basis of V. Then the matrix elements O_{mn} of $\hat{O}$ are defined by

$$O_{nm} = \langle n | \hat{O} | m \rangle$$

From O_{mn}, the operator can be reconstructed as

$$O = \sum_{n,m} O_{nm} \, |n\rangle\langle m| \quad .$$

Application: Harmonic oscillator

$$\hat{H} = \frac{1}{2} \hat{p}^2 + \frac{1}{2} \hat{x}^2 \qquad \begin{pmatrix} m=1 \\ \hbar = 1 \\ \omega = 1 \end{pmatrix}$$

$\hat{H}$ is time independent. Set $\quad \Psi(x,t) = e^{-iEt} \psi(x)$

↳ solve $\quad \hat{H} \psi(x) = E \psi(x)$

Define $\quad \hat{a} = \frac{1}{\sqrt{2}} (\hat{x} + i \hat{p}) = \frac{1}{\sqrt{2}} \left(x + \frac{d}{dx} \right)$

in position space

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}} (\hat{x} - i \hat{p}) = \frac{1}{\sqrt{2}} \left(x - \frac{d}{dx} \right)$$

Can easily verify that $[\hat{a}, \hat{a}^\dagger] = 1$

And $\hat{H} = \hat{a}^+ a + \frac{1}{2} \equiv \hat{N} + \frac{1}{2}$.

Let ψ_v be eigenfunction of $\hat{N}$ with $\hat{N}\psi_v = v\psi_v$, and assume that $(\psi_v, \psi_v) = 1$.

$$v = v(\psi_v, \psi_v) = (\psi_v, \hat{N}\psi_v) = (\hat{a}\psi_v, \hat{a}\psi_v) \geq 0 .$$

$\Rightarrow \quad v \geq 0$.

Now take $v = 0$, which corresponds to a state with energy $\frac{1}{2}$ $(\hbar\omega)$.

$$0 = (\hat{a}\psi_0, \hat{a}\psi_0) \quad \Rightarrow \quad \hat{a}\psi_0 = 0$$

↳ $\hat{a}$ annihilates the ground state (lowest energy).

Solve: $\quad \hat{a}\psi_0 = \left(x + \frac{d}{dx}\right)\psi_0 = 0$

$\Rightarrow \quad \psi_0(x) = \pi^{-\frac{1}{4}} e^{-\frac{1}{2}x^2}$ after normalization.

All other states can be obtained from this ground state:

$$[\hat{N}, \hat{a}^+] = \hat{a}^+$$

$$\Rightarrow \quad \hat{N}(\hat{a}^+\psi_v) = (v+1)\,\hat{a}^+\psi_v$$

$\Rightarrow \quad \psi_{v+1} \equiv \frac{1}{\sqrt{v+1}}\hat{a}^+\psi_v$ is eigenstate with EV $v+1$.

In Dirac notation: $\hat{a}|0\rangle = 0$

$$|1\rangle = a^{+}|0\rangle$$

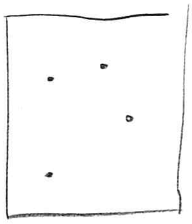
$$|n\rangle = \frac{1}{\sqrt{n!}}\left(\hat{a}^{+}\right)^{n}|0\rangle$$

[Straightforward to show that no other, normalisable solutions exist]

Many body theory (2)

Permutations of particles (2.1)

Consider identical particles in a box



Described by wave function

$$\psi(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n, t) \quad \text{(+ internal d.o.f. + spin)}$$

In QM, identical particles are indistinguishable

↳ Constrains properties of ψ under exchange of particles

More formal:

Let $|n\rangle$ be an orthonormal basis of a 1-particle Hilbert space. $n = (\vec{x}, s, m)$ for example.

Choose $\quad |n_1\rangle_1 \otimes |n_2\rangle_2 \otimes \ldots \otimes |n_N\rangle_N$

as basis of N-particle HS. Notation: $|i\rangle_j$ means particle j is in state $|i\rangle$. Shorthand notation

$$\equiv |1, 2, \ldots, N\rangle$$ [here: Number represents state (i.e. $3 \equiv n_3$), location is particle index]

Introduce exchange operator $\hat{P}_{ij}$. Exchanges eigenvalues of particles i and j!

$$\hat{P}_{ij} |1, \ldots, n_i, \ldots, n_j, \ldots, N\rangle = |1, \ldots, n_j, \ldots, n_i, \ldots, N\rangle$$

[Note: $|1, \ldots, j, \ldots, i, \ldots N\rangle$ means particle i is in state $|n_j\rangle$ etc.]

We have $\left(\hat{P}_{ij}\right)^2 = 1$, therefore $\hat{P}_{ij}$ has EV ± 1.

Indistinguishable particles $\Leftrightarrow [\hat{H}, \hat{P}_{ij}] = 0 \;\; \forall\, i,j \in \{1, \ldots, N\}$.

If $\hat{H}|\psi\rangle = E|\psi\rangle \Rightarrow \hat{H}\hat{P}_{ij}|\psi\rangle = E\hat{P}_{ij}|\psi\rangle$

Properties:

- $\langle \hat{P}_{ij}\phi | \hat{P}_{ij}\psi\rangle = \langle\phi|\psi\rangle$
- $\langle \hat{P}^{+}_{ij}\phi|\psi\rangle \equiv \langle\phi|\hat{P}_{ij}\psi\rangle$
- $\langle\phi|\hat{P}_{ij}\psi\rangle = \langle\hat{P}^{-1}_{ij}\phi|\hat{P}^{-1}_{ij}\hat{P}_{ij}\psi\rangle = \langle\hat{P}^{-1}_{ij}\phi|\psi\rangle$

$\Rightarrow \hat{P}^{+}_{ij} = \hat{P}^{-1}_{ij}$ (unitary) and $\hat{P}^{+}_{ij} = \hat{P}_{ij}$ since $\hat{P}^{-1}_{ij} = \hat{P}_{ij}$ (hermitian).

Note: $\hat{P}_{ij}$, $\hat{P}_{kl}$ don't necessarily commute. The exchange operators generate the group of permutations S_N.

Now, consider one specific $\hat{P}_{ij}$. Since $[\hat{P}_{ij}, \hat{H}] = 0$, we can study simultaneous eigenstates of $\hat{H}$ and $\hat{P}_{ij}$:

We have either

$$\hat{P}_{ij}|1,..i..j..N\rangle = +|1...i..j...N\rangle$$

or

$$\hat{P}_{ij}|1..i..j..N\rangle = -|1..i..j...N\rangle$$

Since i,j are not special, the + or - should hold for all particles. We call

+ Bosons
- Fermions

Spin - Statistics theorem:

Bosons have integer spin, 0 (Higgs), 1 (Photon, W, Z), 2 (graviton?)
Fermions have half integer spin $\frac{1}{2}$ (quarks, leptons, proton), $\frac{3}{2}$ (bound states, gravitino?)

Proof: Relativistic QFT ($\rightarrow$ later)
↳ Corollary: Pauli's principle.

Multi-particle states (2.2)

Def: symmetric operator

$$[\hat{O}, \hat{P}_{ij}] = 0 \quad \forall\, i \neq j$$

$$\Rightarrow \langle \hat{P}_{ij}\phi | \hat{O} | \hat{P}_{ij}\psi\rangle = \langle \phi | \hat{P}_{ij}^{+} \hat{O} \hat{P}_{ij} | \psi\rangle = \langle \phi | \hat{O} | \psi \rangle .$$

Since we can not distinguish identical particles in QM, all physical observables should correspond to symmetric operators!

Consider again N particles with spin s, non interacting.

$$\hat{H} = \sum_{j=1}^{N} \frac{1}{2m} \hat{\vec{P}}_j^{2}$$

mass

One particle states:

$$\psi_{\vec{p},s,m} = \frac{1}{(2\pi)^{3/2}} e^{i\vec{p}\vec{x}} |s,m\rangle$$

component of spin in e.g. z-direction

N-particle system: $(\vec{p}_1, s, m_1) \dots (\vec{p}_N, s, m_N)$ are the N states that appear, although the ordering is arbitrary.

Let us label the particles by their positions x_i. Then

$$\psi_j(i) \equiv \psi_{\vec{p}_j, s, m_j}(x_i)$$

denotes particle i in state j.

A natural ansatz for an N-particle wave function is the product of N 1-particle states. However this does not have the required (anti-) symmetry properties.

Symmetrization (Bosons):

$$|\psi\rangle_B = c_B \sum_{\sigma \in S_N} \psi_{\sigma(1)}(1) \dots \psi_{\sigma(N)}(N) .$$

Anti-symmetrization (Fermions):

$$|\psi\rangle_F = c_F \sum_{\sigma \in S_N} \text{sign}(\sigma)\, \psi_{\sigma(1)}(1) \dots \psi_{\sigma(N)}(N)$$

c_B, c_F are normalization factors $\rightarrow$ exercises

The sign of $\sigma \in S_N$ is $+1$ for even and -1 for odd permutations ($=$ # of exchanges P_{ij})

S_3:	(123)	(132)	(213)	(321)	(312)	(231)
	$+$	$-$	$-$	$-$	$+$	$+$

For fermions, can write $|\psi\rangle_F = c \begin{vmatrix} \psi_1(1) & \dots & \psi_1(N) \\ \vdots & & \vdots \\ \psi_N(1) & \dots & \psi_N(N) \end{vmatrix}$

"Slater determinant".
If two states are equal, then the rows are linearly dependent, and $|\psi\rangle_F = 0$. Pauli principle.

Example: 2 Bosons, different states

$$|\psi\rangle_B = \frac{1}{\sqrt{2}}\left(\psi_1(1)\psi_2(2) + \psi_2(1)\psi_1(2) \right) = \frac{1}{\sqrt{2}}\left(|12\rangle + |21\rangle \right)$$

same state $\quad |\psi_B\rangle = \psi_1(1)\psi_1(2) = |11\rangle$

2 fermions $\quad |\psi\rangle_F = \frac{1}{\sqrt{2}}\left(\psi_1(1)\psi_2(2) - \psi_2(1)\psi_1(2) \right)$

end 18.10.

Fock-space

We now have a framework to describe single particle states and N-particle states. But we might also be interested in processes or systems where particle number changes.

1-particle Hilbert space is vector space over the field $\mathbb{C}$, with an inner product and complete.

$$V_1 = \left\{ |\vec{p}, s, m\rangle \;\middle|\; \vec{p} \in \mathbb{R}^3, 2s \in \mathbb{N}_0, m \in -s, \dots, (s-1), s \right\}$$

It is convenient to define

$$V_0 \equiv \mathbb{C} \quad .$$

Furthermore

$$V_n = \begin{cases} \mathrm{Sym}\,(V_1)^{\otimes n}, & \text{bosons} \\ \mathrm{Asym}\,(V_1)^{\otimes n}, & \text{fermions} \end{cases} ,$$

i.e. the fully symmetrized (antisymmetrized) n-fold tensor product of V_1. Elements of V_n take the form

$$|\psi\rangle_B = \frac{1}{\sqrt{N!\,N_1!\dots N_r!}} \sum_{\sigma \in S_N} \psi_{\sigma(1)}(1) \dots \psi_{\sigma(N)}(N)$$

$$|\psi\rangle_F = \frac{1}{\sqrt{N!}} \sum_{\sigma \in S_N} \mathrm{sign}(\sigma)\, \psi_{\sigma(1)}(1) \dots \psi_{\sigma(N)}(N)$$

The Fock space is defined as the direct sum of all V_n, $n \in \mathbb{N}_0$:

$$V = \bigoplus_{n=0}^{\infty} V_n = V_0 \oplus V_1 \oplus V_2 \oplus \dots$$

Now we need operators that can move states from $V_i \to V_j$

Creation / Annihilation operators (2.4)

Consider first a system with N bosons, such that N_1 bosons are in state $|1\rangle$, N_2 in state $|2\rangle$, i.e.

$$N = N_1 + N_2 + \dots \qquad N_i \in \mathbb{N}_0 .$$

Note: Particles are indistinguishable, only need to know how many are in a given state.

Let
$$\psi_1 = \psi_2 = \dots \psi_{N_1} = |1\rangle$$
$$\psi_{N_1+1} = \dots = \psi_{N_1+N_2} = |2\rangle$$
$$\dots$$

We introduce the <u>occupation number representation</u>:

$$|N_1, N_2, \dots\rangle = \frac{1}{\sqrt{N! N_1! N_2! \dots}} \sum_{\sigma \in S_N} \psi_{\sigma(1)}(1) \dots \psi_{\sigma(N)}(N)$$
$$= \frac{1}{\dots} \sum_{\sigma \in S_N} |n_{\sigma(1)}\rangle_1 \otimes \dots |n_{\sigma(N)}\rangle_N$$

where $|n_i\rangle$ are the states which appear, $i \in \{1 \dots N\}$, and where repetitions are allowed (i.e. $|n_i\rangle = |n_j\rangle$ for $i \neq j$ is allowed)

Normalisation:

$$\langle N_1, N_2 \dots | N_1' N_2' \dots \rangle = \delta_{N_1 N_1'} \delta_{N_2 N_2'} \dots$$

Completeness

$$\sum_{N_1 N_2 \dots} |N_1 N_2 \dots\rangle\langle N_1 N_2 \dots| = \mathbb{1}$$

Finally we introduce <u>creation operators</u> $\hat{a}_j^{\dagger}$ and <u>annihilation operators</u> $\hat{a}_j$:

$\hat{a}_j^{\dagger}$ takes an N - particle state and converts it to an $N+1$ particle state, where the state $|j\rangle$ has its occupation increased by 1 :

$$\hat{a}_j^{\dagger} |N_1, \dots, N_j, \dots\rangle = \sqrt{N_j+1}\, |N_1, \dots, N_j+1, \dots\rangle$$

Of course $|1\rangle$, $|2\rangle$ etc are shorthand notations for states characterised by quantum numbers e.g. $(\vec{p}_1, s, m_1)$, $(\vec{p}_2, s, m_2)$ etc. Can e.g. consider them as independent harmonic oscillators, with energy $N_i E_i$ ...

For the annihilation operator, we find

$$\hat{a}_j |N_1, \dots, N_j, \dots\rangle = \begin{cases} \sqrt{N_j}\, |N_1, \dots, N_j - 1, \rangle & N_j > 0 \\ 0 & N_j = 0 \end{cases}$$

Commutation relations :

$$[\hat{a}_i, \hat{a}_j] = 0 \qquad [\hat{a}_i^{\dagger}, \hat{a}_j^{\dagger}] = 0 \qquad [\hat{a}_i, \hat{a}_j^{\dagger}] = \delta_{ij}$$

Proof : $i \neq j$

$$\hat{a}_i \hat{a}_j |N_1, \dots N_i, N_j, \dots\rangle = \sqrt{N_i}\sqrt{N_j}\, |N_1, \dots, N_i+1, N_j+1, \dots\rangle$$

$$\hat{a}_j \hat{a}_i |N_1, \dots, N_i, N_j, \dots\rangle = \text{same} .$$

Consider $|0\rangle$. Obviously $\hat{a}^+_i \hat{a}_j |0\rangle = 0$
$\hat{a}_i \hat{a}^+_j |0\rangle = \delta_{ij} |0\rangle$.

↳ can show this also for general states $|N_1, \ldots, N_i, \ldots N_j \ldots\rangle$.

All states in V can be obtained from $|0\rangle = |0, 0, \ldots, 0, \ldots\rangle$ by acting with $\hat{a}^+_j$, $\hat{a}^+_k$, … N_j, N_k times.

$$|N_1, N_2, \ldots\rangle = (N_1! \, N_2! \ldots)^{-\frac{1}{2}} (\hat{a}^+_1)^{N_1} (\hat{a}^+_2)^{N_2} \ldots |0\rangle .$$

Particle number operator:

$$\hat{N}_i = \hat{a}^+_i \hat{a}_i$$

Action:

$$\hat{N}_i |N_1, \ldots, N_i, \ldots\rangle = N_i |N_1, \ldots, N_i, \ldots\rangle$$

such that $\langle\psi| \hat{N}_i |\psi\rangle = N_i$

Total particle number:

$$\hat{N} = \sum_i \hat{N}_i .$$

$$\hat{N} |\{N\}\rangle = \left(\sum_i N_i\right) |\psi\rangle = N |\{N\}\rangle$$

Now, consider a system of N non-interacting particles. Let $|j\rangle$ be ES of the single particle Hamiltonian with EV E_j. We can write

$$\hat{H}_j = E_j \hat{a}_j^\dagger \hat{a}_j$$

$\Rightarrow$

$$\hat{H} = \sum_j E_j \hat{a}_j^\dagger \hat{a}_j = \sum_j E_j \hat{N}_j$$

Note: Sum goes over all states!

$$\hat{H} |N_1 N_2 \dots\rangle = \left(\sum E_j N_j\right) |N_1 N_2 \dots\rangle$$

$\hookrightarrow$ energy of state $|j\rangle$ × # particles in state $|j\rangle$

A generic operator that is the sum of N identical 1-particle operators can be written as

$$\hat{O} = \sum_{k=1}^{N} \hat{O}_k$$

particle index.

Let $O_{ij} = {}_k\langle i | \hat{O}_k | j \rangle_k \Rightarrow \hat{O}_k = \sum_{i,j} O_{ij} |i\rangle_k \langle j|_k$

and $$\hat{O} = \sum_{i,j} O_{ij} \sum_{k=1}^{N} |i\rangle_k \langle j|_k$$

Goal: Show that we can write $\hat{O} = \sum_{i,j} O_{ij} \hat{a}_i^\dagger \hat{a}_j$

// end

23.10.18

Q: What is $|0\rangle$?

Evaluate

$$\sum_{k=1}^{N} |i\rangle_k \langle j|_k \; |N_1, \dots N_i \dots N_j \dots\rangle =$$

$$\sum_{k=1}^{N} |i\rangle_k \langle j|_k \frac{1}{\Delta} \sum_{\sigma \in S_N} |n_{\sigma(1)}\rangle_1 \otimes \dots \otimes |n_{\sigma(N)}\rangle_N$$

$$\Delta = \sqrt{N!\, N_1! \dots N_i! \dots N_j!}$$

Since $|i\rangle_k\langle j|_k$ replaces a state $|j\rangle$ with a state $|i\rangle$, we should obtain a state proportional to

$$|N_1 \dots (N_i+1) \dots (N_j-1) \dots\rangle$$

with prefactor N_j, since the state $|j\rangle$ is initially contained N_j times. Taking into account normalisation, we find

$$\sum_{k=1}^{N} |i\rangle_k\langle j|_k \, |N_1 \dots N_i \dots N_j \dots\rangle = \underbrace{N_j \frac{\sqrt{N_i+1}}{\sqrt{N_j}}}_{\sqrt{N_j}\sqrt{N_i+1}} |N_1 \dots (N_i+1) \dots (N_j-1) \dots\rangle$$

$$= \hat{a}_i^{+} \hat{a}_j \, |N_1 \dots, N_i, \dots, N_j \dots\rangle \; .$$

Valid for all states $|N_1 \dots\rangle$, i.e. we find the operator relation

$$\sum_{k=1}^{N} |i\rangle_k \langle j|_k = \hat{a}_i^{+} \hat{a}_j$$

and

$$\hat{O} = \sum_{ij} O_{ij} \, \hat{a}_i^{+} \hat{a}_j$$

Any operator which is a sum of 1-particle operators can be brought into this form.

No interactions yet ! Need (at least) two particle operators

Why? Let us try to describe a scattering process: Two particles, r and s, are scattered from states $|i\rangle$, $|j\rangle$ to $|k\rangle$, $|l\rangle$.

need operator with $|k\rangle_r$, $|l\rangle_s$, $\langle i|_r$, $\langle j|_s$
(+permutations).

Write:

$$\hat{\sigma} = \frac{1}{2} \sum_{r=1}^{N} \sum_{\substack{s=1\\ s\neq r}}^{N} \hat{\sigma}_{rs}$$

$$= \sum_{r=1}^{N-1} \sum_{s=r+1}^{N} \hat{\sigma}_{rs} \qquad \text{since } \hat{\sigma}_{rs} = \hat{\sigma}_{sr} \quad \text{(symmetry bosons!)}$$

Matrix elements

$$\sigma_{ij,kl} = \left(\langle i|_r \otimes \langle j|_s\right) \hat{\sigma}_{rs} \left(|k\rangle_r \otimes |l\rangle_s\right)$$

$$\hat{\sigma} = \frac{1}{2} \sum_{r\neq s} \sum_{ijkl} \hat{\sigma}_{ijkl} \, |i\rangle_r |j\rangle_s \langle k|_r \langle l|_s$$

$$\sum_{r\neq s} |i\rangle_r |j\rangle_s \langle k|_r \langle l|_s = \sum_{r\neq s} |i\rangle_r \langle k|_r |j\rangle_s \langle l|_s$$

$$= \sum_{r,s} |i\rangle_r \langle k|_r \, |j\rangle_s \langle l|_s - \sum_r |i\rangle_r \underbrace{\langle k|_r |j\rangle_r}_{\delta_{kj}} \langle l|_r$$

$$= \hat{a}_i^\dagger \hat{a}_k \hat{a}_j^\dagger \hat{a}_l - \delta_{kj} \hat{a}_i^\dagger \hat{a}_l$$

$$= \hat{a}_i^\dagger \hat{a}_k \hat{a}_j^\dagger \hat{a}_l - \hat{a}_i^\dagger [\hat{a}_k, \hat{a}_j^\dagger] \hat{a}_l = \hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_k \hat{a}_l .$$

$$\Rightarrow \quad \hat{O} = \frac{1}{2} \sum_{ijkl} O_{ijkl} \hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_k \hat{a}_l$$

Note: It is not easy in general to find the matrix elements of $\hat{O}$ wrt single particle states. But a lot can be understood from 1-particle and 2-particle operators

↳ perturbative expansion !

Note 2: Many body physics is hard, even clasically :

3-body problem, chaotic behavior

Virial theorem ...

Fermions (2.4.2)

29.10.18

- Wednesday: no lecture, enjoy the holiday
- Ex: Sheet 2; ex 2 cde, postponed by one week, due Nov 5.
- One comment on exercises: states $|\vec{x}, i\rangle$, completeness rel.

$$\sum_i \int dx \, |x, i\rangle\langle x, i| = \mathbb{1}$$

not $$\sum_i |i\rangle\langle i| + \int dx \, |x\rangle\langle x| .$$

Catch-up from last week:

Occupation number basis

$$|N_1 N_2 \dots \rangle \qquad N_1 \text{ particles in state } |1\rangle, \dots$$

Creation/annihilation op:

$$\hat{a}_j^\dagger |N_1 \dots N_j \dots\rangle = \sqrt{N_j + 1}\, |N_1, \dots N_j + 1 \dots\rangle$$

$$[\hat{a}_i, \hat{a}_j] = [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0 \qquad [\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij} \qquad \text{Bosons}$$

All states from empty state $|00\dots\rangle \equiv |0\rangle \in \mathbb{C}$, $\langle 0|0\rangle = 1$

Note: Harmonic osc. ground state is different $\rightarrow$ not an empty system !
HO does not change particle number

Operators expressed as lin. combinations of products of $\hat{a}_i^+$, $\hat{a}_j$ ...

Hamiltonian $\quad \hat{H} = \sum_j E_j \hat{a}_j^+ \hat{a}_j$

single particle operators $\quad \hat{O} = \sum_{ij} O_{ij} \hat{a}_i^+ \hat{a}_j$

two particle operators $\quad \hat{O} = \frac{1}{2} \sum_{ijkl} O_{ij,kl} \hat{a}_i^+ \hat{a}_j^+ \hat{a}_k \hat{a}_l$.

⋮

Fermions : Pauli-principle $\Rightarrow$ $N_i \in \{0,1\}$

$$\Rightarrow \left(\hat{a}_j^+\right)^2 = 0 \quad \forall j \text{, and } \{\hat{a}_i^+, \hat{a}_j^+\} = 0$$

$$\hat{a}_i^+ |N_1 \dots N_i \dots\rangle = (1-N_i)(-1)^{\sum_{j=1}^{i-1} N_j} |N_1 \dots N_i+1 \dots\rangle .$$

and $\quad \hat{O} = \frac{1}{2} \sum_{ijkl} O_{ij,kl} \hat{a}_i^+ \hat{a}_j^+ \hat{a}_l \hat{a}_k \quad$ (Fermions)

Field operators (2.5)

Consider a change of basis $|n\rangle \to |\tilde{m}\rangle$ (orthonormal sets)

$$|\tilde{m}\rangle = \sum_n |n\rangle\langle n|\tilde{m}\rangle$$

Furthermore $\quad \hat{a}_{\tilde{m}}^+ = \sum_n \langle n|\tilde{m}\rangle \hat{a}_n^+$

$$\hat{a}_{\tilde{m}} = \sum_n \langle \tilde{m}|n\rangle \hat{a}_n$$

Position space representation

Let $|m\rangle$ be a basis of position eigenstates, i.e. $|m\rangle = |\vec{x}\rangle$
(Note: Spin can be added trivially)

$$\langle \vec{x} | i \rangle = \varphi_i(\vec{x}) \quad \text{is the wave fu. of } |i\rangle .$$

We define field operators as operators that generate a particle in state $|\vec{x}\rangle$.

Notation:

$$\hat{\phi}(\vec{x}) = \hat{a}_{\vec{x}} \qquad \hat{\phi}^+(\vec{x}) = \hat{a}^+_{\vec{x}} .$$

We have that

$$\hat{\phi}(\vec{x}) = \hat{a}_{\vec{x}} = \sum_i \langle \vec{x} | i \rangle \hat{a}_i = \sum_i \varphi_i(\vec{x}) \hat{a}_i$$

$$\hat{\phi}^+(\vec{x}) = \sum_i \varphi_i^*(\vec{x}) \hat{a}_i^+$$

$$\hat{\phi}^+(\vec{x}) |0\rangle = |\vec{x}\rangle \quad \left(= |0, \dots, 1_{\vec{x}}, \dots \rangle \right)$$

Commutation relations

B: $[\hat{\phi}(\vec{x}), \hat{\phi}(\vec{y})] = [\hat{\phi}^+(\vec{x}), \hat{\phi}^+(\vec{y})] = 0 \qquad [\hat{\phi}(\vec{x}), \hat{\phi}^+(\vec{y})] = \delta^3(\vec{x} - \vec{y})$

F: $\{\hat{\phi}(\vec{x}), \hat{\phi}(\vec{y})\} = 0 \quad \dots$

Example:

$$[\hat{\phi}(\vec{x}), \hat{\phi}(\vec{y})] = \sum_{ij} \varphi_i(\vec{x}) \varphi_j(\vec{y}) [\hat{a}_i, \hat{a}_j] = 0$$

$$[\hat{\phi}(\vec{x}), \hat{\phi}^+(\vec{y})] = \sum_{ij} \varphi_i(\vec{x}) \varphi_j^*(\vec{y}) [\hat{a}_i, \hat{a}_j^+]$$

$$= \sum_i \varphi_i(\vec{x})\, \varphi_i^*(\vec{y}) = \sum_i \langle \vec{x} | i \rangle \langle i | \vec{y} \rangle$$

$$= (\vec{x}, \vec{y}) = \delta^3(\vec{x} - \vec{y}) \; .$$

Fermions: Exercise.

Now we invert the relation between $\hat{a}_i$ and $\hat{\phi}(\vec{x})$

$$\hat{a}_i = \int d^3x \, \langle i | \vec{x} \rangle \, \hat{a}_{\vec{x}} = \int d^3x \; \varphi_i^*(\vec{x}) \, \hat{\phi}(\vec{x})$$

$$\hat{a}_i^+ = \int d^3x \; \varphi_i(\vec{x}) \, \hat{\phi}^+(\vec{x})$$

Note / Comment: The method of annihilation / creation operators is often called <u>second quantization</u>. While in QM the wave fn. $\varphi_i(\vec{x})$ plays a central role, here it is demoted to a coefficient function. The field operator becomes the central object. (... field theory approach)

Some
Important operators

Kinetic energy:
$$\hat{T} = \sum_{k=1}^{N} \hat{T}_k = \sum_{k=1}^{N} \left(-\frac{1}{2m} \Delta_k \right)$$

$\hat{T}_k$: 1-particle operator

$$= \sum_{ij} T_{ij} \, \hat{a}_i^+ \hat{a}_j \qquad {}_k\langle i | \hat{T}_k | j \rangle_k \quad (\text{same for all } k)$$

Now:
$$\hat{T} = \sum_{ij} \int d^3x \int d^3y \; \varphi_i(\vec{x}) \, \hat{\phi}^+(\vec{x}) \, T_{ij} \, \varphi_j^*(\vec{y}) \, \hat{\phi}(\vec{y})$$

$$= \oint_{\substack{ij \\ xy}} \hat{\phi}^+(\vec{x}) \, \langle \vec{x} | i \rangle \langle i | -\frac{1}{2m} \Delta | j \rangle \langle j | \vec{y} \rangle \, \hat{\phi}(\vec{y})$$

$$= \int d^3x \int d^3y \; \hat{\phi}^+(\vec{x}) \langle \vec{x} | -\frac{1}{2m}\Delta | \vec{y} \rangle \hat{\phi}(\vec{y}) .$$

$$\langle \vec{x} | -\frac{1}{2m}\Delta | \vec{y} \rangle = \int_{\vec{p},\vec{q}} \langle \vec{x} | \vec{p} \rangle \langle \vec{p} | \frac{1}{2m} \hat{\vec{p}}^2 | \vec{q} \rangle \langle \vec{q} | \vec{y} \rangle$$

$$= \int d^3p \int d^3q \; \frac{\vec{q}^2}{2m} \langle \vec{x} | \vec{p} \rangle \underbrace{\langle \vec{p} | \vec{q} \rangle}_{\delta^3(\vec{p}-\vec{q})} \langle \vec{q} | \vec{y} \rangle$$

$$= \int d^3p \; \frac{\vec{p}^2}{2m} \langle \vec{x} | \vec{p} \rangle \langle \vec{p} | \vec{y} \rangle \qquad \langle \vec{x} | \vec{p} \rangle = e^{i\vec{p}\vec{x}}$$

$$= \int d^3p \; \frac{\vec{p}^2}{2m} \frac{\exp(i(\vec{x}-\vec{y})\vec{p})}{(2\pi)^3} = \left(-\frac{1}{2m}\Delta_y\right) \underbrace{\int \frac{d^3p}{(2\pi)^3} \exp(i(\vec{x}-\vec{y})\vec{p})}_{\delta^3(\vec{x}-\vec{y})}$$

$$\Rightarrow \hat{T} = \int d^3x\, d^3y \; \hat{\phi}^+(\vec{x}) \, \delta^3(\vec{x}-\vec{y}) \left(-\frac{1}{2m}\Delta_y\right) \hat{\phi}(y)$$

Note: After partial integration

$$= -\frac{1}{2m} \int d^3x \; \hat{\phi}^+(\vec{x}) \, \Delta \, \hat{\phi}(\vec{x})$$

$$\left[= \frac{1}{2m} \int d^3x \left(\vec{\nabla} \hat{\phi}^+(\vec{x})\right) \left(\vec{\nabla} \hat{\phi}(\vec{x})\right) \right] \text{ partial integration}$$

Note: Derivative acting on field operator directly.

Next case: Non-interacting particles in external potential $U(x)$.

$$\hat{U} = \sum_{k=1}^{N} \hat{U}_k = \sum_{ij} U_{ij} \, \hat{a}_i^+ \hat{a}_j \qquad \text{etc.}$$

$$\hat{U} = \sum_{ij} \int d^3x \int d^3y \; \varphi_i(\vec{x}) \, \hat{\phi}^+(\vec{x}) \, U_{ij} \, \varphi_j^*(\vec{y}) \, \hat{\phi}(\vec{y})$$

one particle op.

$$= \sum_{\substack{ij \\ \vec{x}\vec{y}}} \hat{\phi}^{\dagger}(\vec{x}) \langle \vec{x} | i \rangle \langle i | \hat{U}^{(1)} | j \rangle \langle j | \vec{y} \rangle \hat{\phi}(\vec{y})$$

$$= \int d^3x \; \hat{\phi}^{\dagger}(\vec{x}) \, U^{(1)}(\vec{x}) \, \hat{\phi}(\vec{x})$$

For a two particle operator we find

$$\hat{V} = \frac{1}{2} \int d^3x_1 \int d^3x_2 \; \hat{\phi}^{\dagger}(\vec{x}_1) \hat{\phi}^{\dagger}(\vec{x}_2) \, V^{(2)}(\vec{x}_1, \vec{x}_2) \, \hat{\phi}(\vec{x}_2) \hat{\phi}(\vec{x}_1)$$

Particle number operator:

$$\hat{N} = \sum_i \hat{a}_i^{\dagger} \hat{a}_i = \sum_i \int d^3x \, d^3y \; \varphi_i(\vec{x}) \hat{\phi}^{\dagger}(\vec{x}) \varphi_i^{*}(\vec{y}) \hat{\phi}(\vec{y})$$

$$= \int d^3x \; \hat{\phi}^{\dagger}(\vec{x}) \hat{\phi}(\vec{x})$$

density operator: $\hat{n} = \hat{\phi}^{\dagger}(\vec{x}) \hat{\phi}(\vec{x})$.

Note! In classical physics, $L = T - V$

For a classical field, $\mathcal{L} = \frac{1}{2} \vec{\nabla}\phi^{\dagger} \vec{\nabla}\phi - m^2 \phi^{\dagger}\phi - U(\phi)$.

Now $\phi \to \hat{\phi}$!?

Quantum theory of fields = many body quantum mechanics (+SRT)

Momentum space representation

Write $|\vec{p}\rangle$, and $\langle \vec{p} | i \rangle = \varphi_i(\vec{p})$, $\hat{a}_{\vec{p}}^{\dagger} |0\rangle = |\vec{p}\rangle$.

$$\hat{a}_{\vec{p}}^{\dagger} = \sum_i \varphi_i^{*}(\vec{p}) \, \hat{a}_i^{\dagger} \qquad \hat{a}_i^{\dagger} = \int d^3p \; \varphi_i(\vec{p}) \, \hat{a}_{\vec{p}}^{\dagger} \; .$$

Kinetic energy: $\hat{T} = \sum_{k=1}^{N} \hat{T}_k = \int d^3p \, \frac{p^2}{2m} \, \hat{a}^{+}_{\vec{p}} \hat{a}_{\vec{p}}$

External potential: $U^{(1)}(p,q) \equiv \langle \vec{p} | \hat{U}^{(1)} | \vec{q} \rangle$

$$\Rightarrow \quad \hat{U} = \int d^3p \, d^3q \; U^{(1)}(\vec{p},\vec{q}) \, \hat{a}^{+}_{\vec{p}} \hat{a}_{\vec{q}}$$

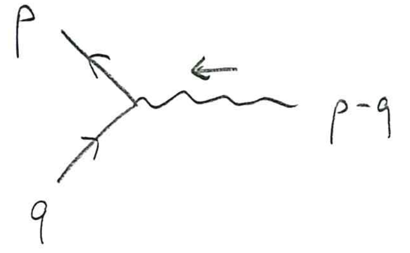
A potential is usually a function of $\vec{x}$ only, with eigenvalues $U^{(1)}(\vec{x})$.
Introduce the fourier transform

$$U^{(1)}(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} \, e^{i\vec{k}\vec{x}} \, \tilde{U}^{(1)}(\vec{k})$$

Can show that $U^{(1)}(\vec{p},\vec{q}) = \frac{1}{(2\pi)^3} \tilde{U}^{(1)}(\vec{p}-\vec{q})$

$$\Rightarrow \quad \hat{U} = \frac{1}{(2\pi)^3} \int d^3p \, d^3q \; \tilde{U}^{(1)}(\vec{p}-\vec{q}) \, \hat{a}^{+}_{\vec{p}} \hat{a}_{\vec{q}}$$

~~Pictorial~~
Graphical representation



For 2-particle interaction, find

$$\hat{U} = \frac{1}{2} \frac{1}{(2\pi)^3} \int d^3p_1 \, d^3p_2 \, d^3q_1 \, d^3q_2 \; \hat{a}^{+}_{\vec{p}_1} \hat{a}^{+}_{\vec{p}_2} \hat{a}_{\vec{q}_2} \hat{a}_{\vec{q}_1} \; \tilde{U}^{(2)}(\vec{p}_1-\vec{q}_1) \, \delta(\vec{p}_1 + \vec{p}_2 - \vec{q}_1 - \vec{q}_2)$$

Heisenberg picture (2.5.3)

Consider a Hamiltonian

$$\hat{H} = \int d^3x \left(\frac{1}{2m} \left(\vec{\nabla}\hat{\phi}^+(x)\right)\left(\vec{\nabla}\hat{\phi}(x)\right) + U^{(1)}(\vec{x})\, \hat{\phi}^+(x)\hat{\phi}(x) \right)$$

$$+ \frac{1}{2}\int d^3x_1\, d^3x_2\, \hat{\phi}^+(x_1)\hat{\phi}^+(x_2)\, U^{(2)}(x_1,x_2)\, \hat{\phi}(x_2)\hat{\phi}(x_1)$$

Convention: $\hat{O}_S$: Schrödinger picture
$\hat{O}_H$: Heisenberg picture

For
Time independent Hamiltonian:

$$\hat{O}_H(x,t) = e^{i\hat{H}t}\, \hat{O}_S(x)\, e^{-i\hat{H}t}$$

in particular $\hat{\phi}^+_H(x,t) = e^{i\hat{H}t}\, \hat{\phi}^+_S(x)\, e^{-i\hat{H}t}$.

Equation of motion:

$$i\frac{\partial}{\partial t}\hat{O}_H = [\hat{O}_H, \hat{H}]$$

For the above $\hat{H}$, find:

$$i\frac{\partial}{\partial t}\hat{\phi}_H(x,t) = \left(-\frac{1}{2m}\Delta + U^{(1)}(x)\right)\hat{\phi}_H(x,t)$$

$$+\int d^3y\, \hat{\phi}_H(y,t)\, U^{(2)}(x,y)\, \hat{\phi}_H(y,t)\, \hat{\phi}_H(x,t)$$

Proof: 1. $\left[\hat\phi_H(x,t), \hat H\right] = e^{i\hat H t}\left[\hat\phi_S(x), \hat H\right] e^{-i\hat H t}$

Furthermore, use

$$\begin{aligned}[\hat A, \hat B \hat C] = \hat A \hat B \hat C - \hat B \hat C \hat A &= \hat A \hat B \hat C - \hat B \hat A \hat C + \hat B \hat A \hat C - \hat B \hat C \hat A \\ &= [\hat A, \hat B]\hat C + \hat B[\hat A, \hat C] \qquad \text{(Bosons)} \\ &= \{\hat A, \hat B\}\hat C - \hat B\{\hat A, \hat C\} \qquad \text{(Fermions)}\end{aligned}$$

Kinetic term: (bosons)

$$\frac{1}{2m}\int d^3y \left[\hat\phi_S(x), (\nabla\hat\phi_S^+(y))(\nabla\hat\phi_S(y))\right] =$$

$$= \frac{1}{2m}\int d^3y \left[\hat\phi_S(x), \nabla\hat\phi_S^+(y)\right]\nabla\hat\phi_S(y) + \frac{1}{2m}\int d^3y \left(\nabla\hat\phi_S^+(y)\right)\left[\hat\phi_S(x), \nabla\hat\phi_S(y)\right]$$

$$= -\frac{1}{2m}\int d^3y \left\{\left[\hat\phi_S(x), \hat\phi_S^+(y)\right]\Delta\hat\phi_S(y) + \left(\Delta\hat\phi_S^+(y)\right)\left[\hat\phi_S(x), \hat\phi_S(y)\right]\right\}$$

$$= -\frac{1}{2m}\int d^3y\, \delta^3(\vec x - y)\,\Delta\hat\phi_S(y) = -\frac{1}{2m}\Delta\hat\phi_S(x) .$$

$U^{(1)}$ term: $\int d^3y\, U^{(1)}(y)\left[\hat\phi_S(x), \hat\phi_S^+(y)\hat\phi_S(y)\right]$

$$= \int d^3y\, U^{(1)}(y)\,\delta(\vec x - \vec y)\,\hat\phi_S(y) = U^{(1)}(x)\,\hat\phi_S(x) .$$

$U^{(2)}$ analogeous

For creation operator, find

$$i\frac{\partial}{\partial t}\hat{\phi}_H^{+}(x,t)=\left(\frac{1}{2m}\Delta-U^{(1)}(x)\right)\hat{\phi}_H^{+}(x,t)-\int d^3y\,\hat{\phi}_H^{+}(x,t)\,\hat{\phi}_H^{+}(y,t)\,U^{(2)}(x,y)\,\hat{\phi}_H(y,t)$$

For Fermions, same result with different intermediate steps.

Density operator: $\hat{n}_H=\hat{\phi}_H^{+}(x,t)\,\hat{\phi}_H(x,t)$.

$$i\frac{\partial}{\partial t}\hat{n}_H(x,t)=i\left[\left(\frac{\partial}{\partial t}\hat{\phi}_H^{+}(x,t)\right)\hat{\phi}(x,t)+\hat{\phi}_H^{+}(x,t)\left(\frac{\partial}{\partial t}\hat{\phi}_H(x,t)\right)\right]$$

Using the results above:

$$i\frac{\partial}{\partial t}\hat{n}_H(x,t)=\frac{1}{2m}\left(\left(\Delta\hat{\phi}_H^{+}(x,t)\right)\hat{\phi}_H(x,t)-\hat{\phi}_H^{+}(x,t)\left(\Delta\hat{\phi}_H(x,t)\right)\right).$$

Introduce

$$\hat{j}_H=\frac{i}{2m}\left(\left(\nabla\hat{\phi}_H^{+}\right)\hat{\phi}_H-\hat{\phi}_H^{+}\left(\nabla\hat{\phi}_H\right)\right)$$

$$\rightarrow\quad i\frac{\partial}{\partial t}\hat{n}_H=-\nabla\hat{j}_H\,.$$

continuity equation

Applications (3)

Bosons. (3.1)

System of N bosons in finite volume V, spin 0.
↳ momenta discretized.

$$\langle \vec{p} | \vec{q} \rangle = \delta_{\vec{p},\vec{q}} \qquad \langle \vec{x} | \vec{p} \rangle = \frac{1}{\sqrt{V}} e^{i\vec{p}\vec{x}}$$

$$|\psi\rangle \equiv |N_1 N_2 \dots \rangle \quad \text{with} \quad N = \sum_{\vec{p}} N_{\vec{p}} .$$

Free bosons (3.1.1)

Expectation value of density operator:

$$n_\psi = \langle \psi | \hat{\phi}^+(\vec{x}) \hat{\phi}(\vec{x}) | \psi \rangle$$

$$= \frac{1}{V} \sum_{\vec{p},\vec{q}} e^{-i(\vec{p}-\vec{q})\vec{x}} \langle \psi | \hat{a}^+_{\vec{p}} \hat{a}_{\vec{q}} | \psi \rangle$$

If $\vec{p} \neq \vec{q}$, $\hat{a}^+_{\vec{p}} \hat{a}_{\vec{q}} |\psi\rangle$ and $|\psi\rangle$ are orthogonal.

$$= \frac{1}{V} \sum_{\vec{p}} \langle \psi | \hat{a}^+_{\vec{p}} \hat{a}_{\vec{p}} | \psi \rangle$$

$$= \frac{1}{V} \sum_{\vec{p}} N_{\vec{p}} = \frac{N}{V} \qquad \text{makes sense}$$

Pair correlation function:

$$n^2_\psi \, g(\vec{x}_1 - \vec{x}_2) = \langle \psi | \hat{\phi}^+(x_1) \hat{\phi}^+(x_2) \hat{\phi}(\vec{x}_2) \hat{\phi}(\vec{x}_1) | \psi$$

(of $|\psi\rangle$)

Expectation value to find one particle at $\vec{x}_1$ and another one at $\vec{x}_2$.

$$\langle\psi|\hat{\phi}_1^{+}\hat{\phi}_2^{+}\hat{\phi}_2\hat{\phi}_1|\psi\rangle = \frac{1}{V^2}\sum_{\substack{\vec{p}_1\vec{p}_2\\ \vec{q}_1\vec{q}_2}} e^{-i(\vec{p}_1-\vec{q}_1)\vec{x}_1}\, e^{-i(\vec{p}_2-\vec{q}_2)\vec{x}_2} \times\langle\psi|\hat{a}^{+}_{p_1}\hat{a}^{+}_{p_2}\hat{a}_{q_2}\hat{a}_{q_1}|\psi\rangle$$

$1 = \vec{x}_1$

Consider two cases: $\vec{p}_1 = \vec{p}_2$ or $\vec{p}_1 \neq \vec{p}_2$

I $\Rightarrow \vec{q}_1 = \vec{q}_2$

II $\Rightarrow$ either $(\vec{p}_1 = \vec{q}_1) + (\vec{p}_2 = \vec{q}_2)$ or $(\vec{p}_1 = \vec{q}_2)$ and $(\vec{p}_2 = \vec{q}_1)$.

$\Rightarrow$ exponential $\rightarrow 1$.

$$\langle\psi|\hat{\phi}^{+}\hat{\phi}^{+}\phi\phi|\psi\rangle = \frac{1}{V^2}\sum_{\vec{p}}\langle\psi|\hat{a}^{+}_{\vec{p}}\hat{a}^{+}_{\vec{p}}\hat{a}_{\vec{p}}\hat{a}_{\vec{p}}|\psi\rangle$$

$$+\frac{1}{V^2}\sum_{\vec{p}_1\neq\vec{p}_2}\langle\psi|\hat{a}^{+}_{\vec{p}_1}\hat{a}^{+}_{\vec{p}_2}\hat{a}_{p_2}\hat{a}_{p_1}|\psi\rangle$$

$$+\frac{1}{V^2}\sum_{p_1\neq p_2}\langle\psi|\hat{a}^{+}_{p_1}\hat{a}^{+}_{p_2}\hat{a}_{p_1}\hat{a}_{p_2}|\psi\rangle\, e^{-i(\vec{p}_1-\vec{p}_2)(\vec{x}_1-\vec{x}_2)}$$

$$= \frac{1}{V^2}\sum_{\vec{p}} N_{\vec{p}}(N_{\vec{p}}-1) + \frac{1}{V^2}\sum_{\vec{p}_1\neq\vec{p}_2} N_{\vec{p}_1}N_{\vec{p}_2}.$$

$$+\frac{1}{V^2}\sum_{\vec{p}_1\neq\vec{p}_2} e^{i(\vec{p}_1-\vec{p}_2)(\vec{x}_1-\vec{x}_2)} N_{\vec{p}_1}N_{\vec{p}_2}.$$

$$= -\frac{1}{V^2}\sum_{\vec{p}} N_{\vec{p}}(N_{\vec{p}}+1) + \frac{1}{V^2}\Big(\sum_{\vec{p}} N_{\vec{p}}\Big)^2 + \frac{1}{V^2}\Big|\sum_{\vec{p}} e^{-i\vec{p}(\vec{x}_1-\vec{x}_2)} N_{\vec{p}}\Big|^2$$

Side step: $\sum_{\vec{p}_1\vec{p}_2} e^{i(\vec{p}_1-\vec{p}_2)(\vec{x}_1-\vec{x}_2)} N_{\vec{p}_1} N_{\vec{p}_2}$

$$= \left(\sum_{\vec{p}_1} e^{-i\vec{p}_1(\vec{x}_1-\vec{x}_2)} N_{\vec{p}_1}\right)\left(\sum_{\vec{p}_2} e^{+i\vec{p}_2(\vec{x}_1-\vec{x}_2)} N_{\vec{p}_2}\right)$$

$$= \left|\sum_{\vec{p}} e^{-i\vec{p}(\vec{x}_1-\vec{x}_2)} N_p\right|^2$$

$$= \frac{N^2}{V^2} - \frac{1}{V^2}\sum_{\vec{p}} N_{\vec{p}}(N_{\vec{p}}+1) + \frac{1}{V^2}\left|\sum_{\vec{p}} e^{-i\vec{p}(\vec{x}_1-\vec{x}_2)} N_{\vec{p}}\right|^2 .$$

Special cases:

1) All bosons in same state $|\vec{p}_0\rangle$: $N_{\vec{p}} = N\,\delta_{\vec{p},\vec{p}_0}$

$$n_\psi^2\, g(\vec{x}_1,\vec{x}_2) = \langle\psi|\hat{\phi}^+(\vec{x}_1)\,\hat{\phi}^+(\vec{x}_2)\,\hat{\phi}(\vec{x}_2)\,\hat{\phi}(\vec{x}_1)\rangle$$

$$= \frac{1}{V^2}\left(N^2+N^2-N(N+1)\right) = \frac{N(N-1)}{V^2}$$

Interpretation: Chance of finding 1^{st} particle is $\frac{N}{V}$, second particle is $\frac{N-1}{V}$, independent of $\vec{x}_1, \vec{x}_2$.

2) Gaussian distribution of momenta:

$$n_{\vec{p}} = \frac{N}{(\sqrt{2\pi}\,\sigma)^3}\exp\left(-\frac{(\vec{p}-\vec{p}_0)^2}{2\sigma^2}\right)$$

Note: Switch back to continuous momenta, i.e. large V limit

$$\rightarrow \int d^3p\, n_{\vec{p}} = N .$$

Have: $\sum_{\vec p} N_{\vec p} = \sum_{\vec p} (\Delta p)^3 \frac{N_{\vec p}}{(\Delta \vec p)^3} = \int d^3p \, n_{\vec p}$

In particular:

$$\sum_{\vec p} N_{\vec p}^2 = \sum_{\vec p} (\Delta p)^3 \left(\frac{N_{\vec p}}{(\Delta p)^3}\right)^2 (\Delta p)^3$$

$$\rightarrow \lim_{\Delta p \to 0} \int d^3p \, n_{\vec p}^2 \, (\Delta p)^3 = 0 .$$

$$\Rightarrow n_\psi^2 \, g(\vec x_1, \vec x_2) = \frac{N^2}{V^2} - \frac{1}{V^2} \sum_{\vec p} N_{\vec p}(N_{\vec p}+1) + \frac{1}{V^2} \left| \sum_{\vec p} e^{-i\vec p(\vec x_1 - \vec x_2)} N_{\vec p} \right|^2$$

$$\longrightarrow n_\psi^2 - \frac{1}{V^2} \int d^3\vec p \, n_{\vec p} + \frac{1}{V^2} \left| \int d^3p \, e^{-i\vec p(\vec x_1 - \vec x_2)} n_{\vec p} \right|^2$$

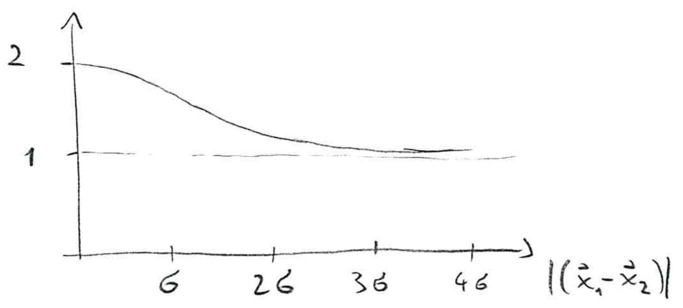
$$\hookrightarrow \frac{N}{V^2} = \frac{n_\psi}{V}$$

Have

$$\frac{N}{(\sqrt{2\pi}\,\sigma)^3} \int d^3p \, e^{-\frac{(\vec p - \vec p_0)^2}{2\sigma^2} - i\vec p(\vec x_1 - \vec x_2)} = N e^{-\frac{1}{2}\sigma(\vec x_1 - \vec x_2)^2} e^{-i\vec p_0(\vec x_1 - \vec x_2)}$$

$$\Rightarrow n_\psi^2 \, g(\vec x_1, \vec x_2) = n^2 \left(1 + e^{-\sigma^2(\vec x_1 - \vec x_2)^2}\right) + \mathcal{O}\left(\frac{1}{V}\right) .$$

$g(\vec x_1, \vec x_2) = 1 + \exp\left(-\sigma^2(\vec x_1 - \vec x_2)^2\right)$ in $V \to \infty$ limit.



Bosons more likely to be in same place.

Hanbury-Brown, Twiss effect

Bunching of photons observed in PMT's, from distant star.
$\hookrightarrow$ quantum optics.

Weakly interacting, dilute bosons, low temperature (3.1.2)

$\hookrightarrow$ neglect 3-particle and higher interactions

Box with volume V, assume $V(\vec{x}_1, \vec{x}_2) = V(\vec{x}_1 - \vec{x}_2)$.

$$\hat{H} = \sum_{\vec{p}} \frac{\vec{p}^2}{2m} \hat{a}^\dagger_{\vec{p}} \hat{a}_{\vec{p}} + \frac{1}{2V} \sum_{\substack{\vec{p}_1 \vec{p}_2 \\ \vec{q}_1 \vec{q}_2}} \delta_{\vec{p}_1+\vec{p}_2, \vec{q}_1+\vec{q}_2} \tilde{V}^{(2)}(\vec{p}_1 - \vec{q}_1) \hat{a}^\dagger_{p_1} \hat{a}^\dagger_{p_2} \hat{a}_{q_2} \hat{a}_{q_1}$$

Expect many bosons in ground state, $\vec{p} = \vec{0}$. Note occupation # in ground state.

$$|\psi\rangle = |N_0, N_1, N_2, \ldots\rangle$$

At very low temps, expect $N_0 \approx N$, $N_{j \neq 0} \ll N$. Neglect interactions of $i, j \neq 0$ states. Focus on 0-0 and 0-i interactions.

Introduce $$\sum_{\vec{p}}{}' = \sum_{\vec{p} \neq \vec{0}}, \quad \hat{a}^\dagger_0 = \hat{a}^\dagger_{\vec{0}}, \quad \tilde{V}^{(2)}_0 \equiv \tilde{V}^{(2)}(\vec{0})$$

$$\hat{H} = \sum_{\vec{p}} \frac{p^2}{2m} \hat{a}^\dagger_{\vec{p}} \hat{a}_{\vec{p}} + \frac{1}{2V} \tilde{V}^{(2)}_0 \hat{a}^\dagger_0 \hat{a}^\dagger_0 \hat{a}_0 \hat{a}_0 + \frac{1}{V} \sum_{\vec{p}}{}' \tilde{V}^{(2)}_0 \hat{a}^\dagger_{\vec{p}} \hat{a}^\dagger_0 \hat{a}_0 \hat{a}_{\vec{p}}$$
$$+ \frac{1}{V} \sum_{\vec{p}}{}' \tilde{V}^{(2)}(\vec{p}) \hat{a}^\dagger_{\vec{p}} \hat{a}^\dagger_0 \hat{a}_{\vec{p}} \hat{a}_0 + \frac{1}{2V} \sum_{\vec{p}}{}' \tilde{V}^{(2)}(\vec{p}) \hat{a}^\dagger_{\vec{p}} \hat{a}^\dagger_{-\vec{p}} \hat{a}_0 \hat{a}_0$$
$$+ \frac{1}{2V} \sum_{\vec{p}}{}' \tilde{V}^{(2)}(\vec{p}) \hat{a}^\dagger_0 \hat{a}^\dagger_0 \hat{a}_{\vec{p}} \hat{a}_{-\vec{p}} + \mathcal{O}(\hat{a}^3_{\vec{p}})$$

$$= \sum_{\vec{p}} \frac{p^2}{2m} \hat{a}_p^\dagger \hat{a}_p + \frac{1}{2V} \tilde{U}_0^{(2)} \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_0 \hat{a}_0 + \frac{1}{V} \sum_{\vec{p}}{}' \left(\tilde{U}_0^{(2)} + \tilde{U}^{(2)}(\vec{p}) \right) \hat{a}_0^\dagger \hat{a}_0 \hat{a}_{\vec{p}}^\dagger \hat{a}_{\vec{p}}$$

$$+ \frac{1}{2V} \sum_{\vec{p}} \tilde{U}^{(2)}(\vec{p}) \left(\hat{a}_{\vec{p}}^\dagger \hat{a}_{-\vec{p}}^\dagger \hat{a}_0 \hat{a}_0 + \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_{\vec{p}} \hat{a}_{-\vec{p}} \right) + \mathcal{O}(3) .$$

Further approximations: If $N \approx N_0 \sim \mathcal{O}(10^{23})$, then N_0 and N_0+1 don't differ by much, and the same for $|N_0 \ldots\rangle$, $|N_0+1 \ldots\rangle$, $|N_0-1 \ldots\rangle$.

$$\Rightarrow \quad \hat{a}_0^\dagger |N_0 \ldots\rangle = \sqrt{N_0+1}\, |N_0+1 \ldots\rangle \approx \sqrt{N_0}\, |N_0 \ldots\rangle$$
$$\hat{a}_0 |N_0 \ldots\rangle \approx \sqrt{N_0}\, |N_0 \ldots\rangle$$

Can replace $\hat{a}_0$, $\hat{a}_0^\dagger$ by $\sqrt{N_0}$, i.e. a number! Find

$$\hat{H} = \sum_{\vec{p}}{}' \frac{p^2}{2m} \hat{a}_{\vec{p}}^\dagger \hat{a}_{\vec{p}} + \frac{N_0^2}{2V} \tilde{U}_0^{(2)} + \frac{N_0}{V} \sum_{\vec{p}}{}' \left(\tilde{U}_0^{(2)} + \tilde{U}^{(2)}(\vec{p}) \right) \hat{a}_{\vec{p}}^\dagger \hat{a}_{\vec{p}}$$
$$+ \frac{N_0}{2V} \sum_{\vec{p}}{}' \tilde{U}^{(2)}(\vec{p}) \left(\hat{a}_{\vec{p}}^\dagger \hat{a}_{-\vec{p}}^\dagger + \hat{a}_{\vec{p}} \hat{a}_{-\vec{p}} \right)$$

Now use $N = N_0 + \sum_{\vec{p}}{}' \hat{a}_{\vec{p}}^\dagger \hat{a}_p$, throwing away $\mathcal{O}(\hat{a}_{\vec{p}}^3)$ and higher terms

$$\rightarrow \; = \sum_{\vec{p}}{}' \frac{p^2}{2m} \hat{a}_{\vec{p}}^\dagger \hat{a}_{\vec{p}} + \frac{N^2}{2V} \tilde{U}_0^{(2)} + \frac{N}{2V} \sum_{\vec{p}}{}' \tilde{U}^{(2)}(\vec{p}) \left(2\hat{a}_p^\dagger \hat{a}_{\vec{p}} + \hat{a}_p^\dagger \hat{a}_{-\vec{p}}^\dagger + \hat{a}_{\vec{p}} \hat{a}_{-\vec{p}} \right)$$

To diagonalise $\hat{H}$, perform Bogoliubov transformation

[Goal: $H = \text{const} + \sum_{\vec{p}}{}' C(\vec{p})\, \hat{a}_{\vec{p}}^\dagger \hat{a}_{\vec{p}}$]

<u>Define:</u>

$$\hat{b}_{\vec{p}} = u_{\vec{p}} \hat{a}_{\vec{p}} + v_{\vec{p}} \hat{a}_{-\vec{p}}^\dagger$$

$$\hat{b}_{\vec{p}}^\dagger = u_{\vec{p}}^* \hat{a}_{\vec{p}}^\dagger + v_{\vec{p}}^* \hat{a}_{-\vec{p}}$$