Second example: Interaction of electrons via phonon exchange.

Initial state: $\vec{q}_1, \uparrow$ and $\vec{q}_2, \downarrow$ $\quad |i\rangle = \hat{a}^+_{\vec{q}_1\uparrow} \hat{a}^+_{\vec{q}_2\downarrow} |0\rangle$

Final state: $\vec{p}_1, \uparrow$ and $\vec{p}_2, \downarrow$ $\quad |f\rangle = \hat{a}^+_{\vec{p}_1\uparrow} \hat{a}^+_{\vec{p}_2\downarrow} |0\rangle$

$$E_i = E^e_{\vec{q}_1} + E^e_{\vec{q}_2} \qquad E_f = E^e_{\vec{p}_1} + E^e_{\vec{p}_2}$$

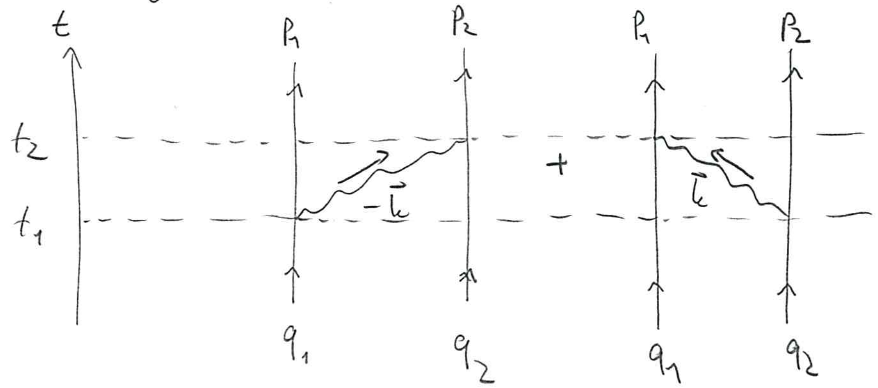
First order perturbation theory: $\langle f | \hat{H}_1 | i \rangle = 0$.

Nontrivial contribution at second order:

$$S^{(2)}_{fi} = (-i)^2 \sum_j \int_{t_i}^{t_f} dt_2 \int_{t_i}^{t_2} dt_1 \, e^{-iE_f(t_f - t_2)} \langle f | \hat{H}_1 | j \rangle e^{-iE_j(t_j - t_i)} \langle j | \hat{H}_1 | i \rangle e^{-iE_i(t_1 - t_i)}$$

$$= -\exp(-i(E_f t_f - E_i t_i)) \sum_j \int_{t_i}^{t_f} dt_2 \int_{t_i}^{t_2} dt_1 \, e^{i(E_f - E_j)t_2} \langle f | \hat{H}_1 | j \rangle e^{i(E_j - E_i)t_1} \langle j | \hat{H}_1 | i \rangle$$

Feynman diagrams:



Only states with intermediate phonons contribute!

Intermediate states $|j\rangle$ that contribute:

$$|j_a\rangle = \hat{a}^{\dagger}_{\vec{p}_1,\uparrow}\,\hat{a}^{\dagger}_{\vec{q}_2,\downarrow}\,\hat{b}^{\dagger}_{-\vec{k}}|0\rangle \quad \text{with } E_{j_a} = E^e_{\vec{p}_1} + E^e_{\vec{q}_2} + E^\gamma_{-\vec{k}}$$

$$|j_b\rangle = \hat{a}^{\dagger}_{\vec{q}_1\uparrow}\,\hat{a}^{\dagger}_{\vec{p}_2\downarrow}\,\hat{b}^{\dagger}_{\vec{k}}|0\rangle \quad \text{with } E_{j_b} = E^e_{\vec{q}_1} + E^e_{\vec{p}_2} + E^\gamma_{\vec{k}}$$

Check: $\langle j_a|\hat{H}_1|i\rangle = \langle 0|\hat{b}_{-\vec{k}}\,\hat{a}_{\vec{q}_2\downarrow}\,\hat{a}_{\vec{p}_1\uparrow}\frac{1}{\sqrt{V}}\sum_{\vec{p}\vec{q}\vec{k}'\sigma} g_{\vec{q},\vec{k}}\,\delta_{\vec{p},\vec{q}+\vec{k}}\,\hat{a}^{\dagger}_{\vec{p}\sigma}\,\hat{a}_{\vec{q},\sigma}(\hat{b}^{\dagger}_{-\vec{k}'}+\hat{b}_{\vec{k}'})$

$$\hat{a}^{\dagger}_{\vec{q}_1\uparrow}\,\hat{a}^{\dagger}_{\vec{q}_2\downarrow}|0\rangle$$

nonzero with $\vec{k}' = k$, $\vec{q} = \vec{q}_1$, $\vec{p} = \vec{p}_1$, $\sigma = \uparrow$, and $p_1 = \vec{q}_1 + \vec{k}$

similar for $\langle f|\hat{H}_1|j_a\rangle$ and $|j_b\rangle$ ...

The M_{fj}, M_{ji} reduce to factors of $g_{\vec{q},\vec{k}}$ and δ...

$\Rightarrow$

$$S^{(2)}_{fi} = -\frac{e^{-iE_f t_f - E_i t_i}}{V}\, g_{\vec{q}_1,\vec{k}}\, g_{\vec{q}_2,-\vec{k}}\, \delta_{\vec{p}_1+\vec{p}_2,\,\vec{q}_1+\vec{q}_2} \times$$

$$\int_{t_i}^{t_f} dt_2 \int_{t_i}^{t_2} dt_1 \left(e^{i(E_f - E_{ja})t_2}\, e^{i(E_{ja}-E_i)t_1} + e^{i(E_f - E_{jb})t_2}\, e^{i(E_{jb}-E_i)t_1} \right)$$

Now some work remains to do the integrals. Take the limit $t_i \to -\infty$, $t_f \to +\infty$. Shift $E_{ja} \to E_{ja} - i\delta$, $E_{jb} \to E_{jb} - i\delta$, so the boundary terms at $t = t_i$ vanish.
(Alternative: Imagine that $\hat{H}_1$ is switched off outside of interaction region)

$$\Rightarrow \quad S_{fi}^{(2)} =$$

$$= \frac{i}{V} e^{-i\varphi_{fi}} g_{\vec{q}_1,\vec{k}}\, g_{\vec{q}_2,-\vec{k}}\, \delta_{\vec{p}\ldots} \left(\frac{1}{E_{ja} - E_i} + \frac{1}{E_{jb} - E_i} \right) \int_{t_i}^{t_f} dt_2\, e^{i(E_f - E_i)t_2}$$

$$= \frac{2\pi i}{V} e^{-i\varphi_{fi}} g_{\vec{q}_1,\vec{k}}\, g_{\vec{q}_2,-\vec{k}}\, \delta_{\vec{p}\ldots} \frac{E_{ja} + E_{jb} - 2E_i}{(E_{ja} - E_i)(E_{jb} - E_i)} \delta(E_f - E_i)$$

Finally, use $E_f = E_i$ and $E^{\gamma}_{\vec{k}} = E^{\gamma}_{-\vec{k}}$ to find

$$S_{fi}^{(2)} = -\frac{2\pi i}{V} e^{-i\varphi_{fi}} g_{\vec{q}_1,\vec{k}}\, g_{\vec{q}_2,-\vec{k}} \frac{2E^{\gamma}_{\vec{k}}}{(E^{e}_{\vec{p}_1} - E^{e}_{\vec{q}_1})^2 - (E^{\gamma}_{\vec{k}})^2} \delta_{\vec{p}_1+\vec{p}_2,\vec{q}_1+\vec{q}_2}\, \delta(E_i - E_f)$$

$\hookrightarrow$ this is what we used in the chapter about superconductivity.

Note: $\hat{H}_1$ is the simplest interaction term that describes electrons interacting with the atomic lattice. The $g_{\vec{k},\vec{q}}$ could in principle be calculated from first principles, however this is virtually impossible in practice (10^{23} particles).
Instead $g_{\vec{q},\vec{k}}$ can be measured.
Systematic improvements: higher orders of S_{fi} and more terms in $\hat{H}_{int}$. (EFT)
(QFT: $\hat{H}_{int}$ constrained by symmetry)

Effective theories:

What if we are not interested in the phonons? Can we find an effective theory of electrons only, that agrees with the e-γ theory up to some order in perturbation theory?

Ansatz:

$$\hat{H}_{eff} = \sum_{\vec{p},\sigma} E^e_{\vec{p}}\, \hat{a}^\dagger_{\vec{p},\sigma}\, \hat{a}_{\vec{p},\sigma}$$

$$+ \frac{1}{2V} \sum_{\substack{\vec{p}_1 \vec{p}_2 \vec{q}_1 \vec{q}_2 \\ \sigma_1 \sigma_2}} \delta_{\vec{p}_1+\vec{p}_2, \vec{q}_1+\vec{q}_2}\, \tilde{U}_\gamma\, \hat{a}^\dagger_{\vec{p}_1\sigma_1}\, \hat{a}^\dagger_{\vec{p}_2\sigma_2}\, \hat{a}_{\vec{q}_2\sigma_2}\, \hat{a}_{\vec{q}_1\sigma_1}$$

Consider same process: $|i\rangle = \hat{a}^\dagger_{\vec{q}_1\uparrow}\, \hat{a}^\dagger_{\vec{q}_2\downarrow}|0\rangle$

$|f\rangle = \hat{a}^\dagger_{\vec{p}_1\uparrow}\, \hat{a}^\dagger_{\vec{p}_2\downarrow}|0\rangle$

$$S^{(2,eff)}_{fi} = (-i)\int_{t_i}^{t_f} dt_1\, e^{-iE_f(t_f-t_1)} \langle f|\hat{H}_{eff}|i\rangle\, e^{-iE_i(t_1-t_i)}$$

$$= (-i)\, e^{-i\varphi_{fi}} \int_{t_i}^{t_f} dt_1\, e^{i(E_f-E_i)t_1} \langle f|\hat{H}_{eff}|i\rangle$$

$$= -\frac{2\pi i}{V}\, e^{-i\varphi_{fi}}\, \tilde{U}_\gamma\, \delta_{\vec{p}_1+\vec{p}_2, \vec{q}_1+\vec{q}_2}\, \delta(E_f-E_i)$$

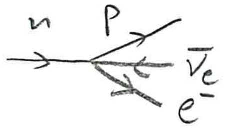
Compare with $S^{(2)}_{fi}$ from before, find

$$\tilde{U}_\gamma = 2\, g_{\vec{q}_1, \vec{k}}\, g_{\vec{q}_2, -\vec{k}} \frac{E^\gamma_{\vec{k}}}{(E^e_{\vec{p}_1} - E^e_{\vec{q}_1})^2 - (E^\gamma_{\vec{k}})^2}$$

Predictions of $\hat{H}_{eff}$ agree with full theory up to $\mathcal{O}(g^2)$.

Uses: • Simpler calculation, only takes into account relevant degrees of freedom

• Sometimes the "full" theory is not known, e.g. β-decay



$\sim$ 1930s

$\sim$ 1970s

Modern viewpoint: All theories are effective theories valid up to some scale Λ, accurately describe dynamics of particles with $m \ll \Lambda$, $E \ll \Lambda$.

Increasing the energy might reveal a new, more fundamental theory

$\swarrow$ $\searrow$ LHC

DM ...

Relativistic QM

4.0 Notation, special relativity

So far, considered systems with $|\vec{v}| \ll 1$. $\Rightarrow$ Ok to use equations that are not Lorentz-invariant

$$E = mc^2 = \gamma m_0 c^2$$

$$= m_0 c^2 \frac{1}{\sqrt{1-v^2/c^2}} = m_0 c^2 \left(1 + \frac{1}{2}\frac{v^2}{c^2} + \mathcal{O}\left(\left(\frac{v}{c}\right)^4\right)\right)$$

$$= \underbrace{m_0 c^2}_{\text{rest energy}} + \underbrace{\frac{1}{2} m_0 v^2}_{\text{kinetic energy}} + \underbrace{\mathcal{O}\left(\left(\frac{v}{c}\right)^4\right)}_{\text{relativistic correction}} .$$

From now on, $m_0 \rightarrow m$.

Now, extension of QM to <u>free</u> relativistic systems. Interacting systems $\rightarrow$ QFT

SRT: Space and time connected via Lorentz transformations.
Notation:

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z) = (ct, \vec{x})$$

$$p^\mu = (p^0, p^1, p^2, p^3) = \left(\frac{E}{c}, \vec{p}\right)$$

Note: $u^\mu = \dfrac{dx^\mu}{ds}$ $\qquad ds = c\,dt\sqrt{1-\dfrac{v^2}{c^2}}$ $\qquad v^i = \dfrac{dx^i}{dt}$

$u^\mu u_\mu = 1 \qquad u^\mu = (\gamma, \gamma\vec{v})$

$\Rightarrow$ 4-momentum: $\quad p^\mu = m\frac{dx^\mu}{d\tau} = mc\frac{dx^\mu}{ds} = mc\,u^\mu \qquad (ds = c\,d\tau)$

$$= (\gamma mc, \gamma m\vec{v})$$

and $\quad p_\mu p^\mu = m^2c^2 u_\mu u^\mu = m^2c^2 = \left(\frac{E}{c}\right)^2 - \vec{p}^{\,2}$

$= 0$ for massless particles.

Metric tensor:

$$g_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

Such that the invariant line element can be written as

$$x_\mu x^\mu = g_{\mu\nu}x^\mu x^\nu = c^2t^2 - x^2 - y^2 - z^2 = c^2t^2 - \vec{x}^{\,2}.$$

$$\Rightarrow \quad x_\mu = g_{\mu\nu}x^\nu = (ct, -\vec{x})$$

Note: Always work with scalar quantities ($p_\mu p^\mu$ etc), never worry about signs.

Lorentz transformations leave line element invariant:

$$x^\mu \rightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

Such that $\quad x'_\mu x'^\mu = x_\mu x^\mu$

$$\Leftrightarrow \quad \Lambda^\mu{}_\sigma \, g_{\mu\nu} \, \Lambda^\nu{}_\tau = g_{\sigma\tau}$$

$$\left(= g_{\mu\nu} \Lambda^\mu{}_\sigma \Lambda^\nu{}_\tau \dots\right)$$

The set of all such transformations forms the Lorentz group

Four components: $\quad \det(\Lambda) = \pm 1 \quad \Lambda^0{}_0 \underset{(\leq -1)}{\geq 1}$

To write a Lorentz invariant theory, write only expressions that are either invariant (e.g. $g_{\mu}b^{\mu}$) or with well defined transformation properties: x^μ, $g_{\mu\nu}$, $T^{\mu\nu}$.

<u>Note:</u>

$$\frac{\partial}{\partial x_\mu} x^2 = \frac{\partial}{\partial x_\mu}(x_\nu x^\nu) = 2\left(\frac{\partial x_\nu}{\partial x_\mu}\right) x^\nu = 2x^\nu$$

$$\Rightarrow \quad \frac{\partial}{\partial x_\mu} \equiv \partial^\mu \quad \text{(transforms as kontravariant tensor)}$$

Klein Gordon eqn

Clearly the Schrödinger eqn. does not have well defined LT properties:

$$i \frac{\partial}{\partial t} |\psi\rangle = \hat{H} |\psi\rangle = \frac{1}{2m} \Delta |\psi\rangle$$

1st derivative in time

2nd derivative in spatial coords

Classically, non-relativistic:

$$E = \frac{1}{2m} \vec{p}^{\,2}$$

QM: $E \to i\frac{\partial}{\partial t}$, $\vec{p} \to -i\vec{\nabla}$

$$\Rightarrow \quad i\frac{\partial}{\partial t}|\psi\rangle = -\frac{1}{2m}\vec{\nabla}^2|\psi\rangle$$ Free Schrödinger eqn.

SRT: $E^2 = m^2c^4 + \vec{p}^{\,2}c^2$

same "quantization":

$$-\frac{\partial^2}{\partial t^2}|\psi\rangle = m^2c^4|\psi\rangle - c^2\Delta|\psi\rangle$$

$$\Leftrightarrow \quad \left(\frac{\partial^2}{c^2\partial t^2} - \Delta + m^2c^2\right)|\psi\rangle = 0$$

$\boxed{\text{Set } c = 1}$

Introduce: $$\Box = \frac{\partial^2}{\partial t^2} - \Delta = \frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x^\mu} \equiv \partial^\mu \partial_\mu$$

d'Alembertian

$$\Rightarrow \quad (\Box + m^2)|\psi\rangle = 0$$ Klein Gordon equation

Obviously Lorentz invariant.

Now, we would like to solve the KG eqn and interpret the solutions as free states of the relativistic theory. We immediately encounter some problems:

1. Solutions. Set $E_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$

Can easily verify that

$$|\psi_1\rangle = \exp(-i(E_{\vec{p}}t - \vec{p}\vec{x}))$$

$$|\psi_2\rangle = \exp(-i(-E_{\vec{p}}t - \vec{p}\vec{x})) = \exp(i(E_{\vec{p}}t + \vec{p}\vec{x}))$$

solve KG.

$$(m^2 + \Box)|\psi_1\rangle = \left(\frac{\partial^2}{\partial t^2} - \Delta + m^2\right) e^{-i(E_{\vec{p}}t - \vec{p}\vec{x})} = \left(-E_{\vec{p}}^2 + \vec{p}^2 + m^2\right)|\psi_1\rangle = 0$$

$|\psi_1\rangle$ corresponds to a plane wave propagating in $\vec{p}$ direction, with positive energy. However $|\psi_2\rangle$ seems to have negative energy, $(\hat{E} = i\frac{\partial}{\partial t})$. This is due to $E^2 = \hat{H}^2 = -\frac{\partial^2}{\partial t^2}$ in KG, i.e. if E is a solution, so is $-E$, for fixed $\vec{p}$.

<u>Anti-particle!?</u>

Or, interpretation of particle with positive energy moving backwards in time.

Theory admits no ground state, can always add negative energy particles to lower total energy.

2. Probability

In QM, $\rho(\vec{x},t) = \psi^*(\vec{x},t)\,\psi(\vec{x},t)$ is interpreted as probability to find a particle at $\vec{x}$ at time t.

Furthermore with $\vec{j}(\vec{x},t) = \frac{1}{2mi}\left(\psi^* \vec{\nabla}\psi - (\nabla\psi^*)\psi\right)$

the continuity equation holds:

$$\frac{\partial}{\partial t}\rho(\vec{x},t) + \vec{\nabla}\vec{j}(\vec{x},t) = 0 .$$

Relativistic QM:

$$\text{define} \quad j^\mu = \frac{i}{2m}\left(\psi^* \partial^\mu \psi - (\partial^\mu \psi^*)\psi\right)$$

$$\text{Find} \quad \partial_\mu j^\mu = \frac{i}{2m}\left((\partial_\mu \psi^*)\partial^\mu \psi + \psi^* \Box \psi - (\Box\psi^*)\psi - (\partial^\mu\psi^*)\partial_\mu \psi\right)$$
$$= 0$$

$$\text{However} \quad j^0 = \rho = \frac{i}{2m}\left(\psi^* \frac{\partial}{\partial t}\psi - \left(\frac{\partial}{\partial t}\psi^*\right)\psi\right)$$

is not positive definite, so can not be interpreted as probability.

Meaningful interpretation as charge density, with anti-particles carrying opposite charge.

Historically, this led to searches for other relativistic wave eqn, e.g. the Dirac equation.

Later it was realized that quantizing $\phi(\vec{x},t)$ as field operator allows a consistent interpretation of the KG eqn.

Next:
- non-rel limit
- Lagrange formalism

Non-relativistic limit

Rewrite KG as

$$\frac{\partial^2}{\partial t^2} \psi(x) = \left(\Delta - m^2\right) \psi(x)$$

2nd order ODE can be rewritten as two 1st order ODEs:

$$\frac{\partial}{\partial t} \begin{pmatrix} \psi \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \Delta - m^2 & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \dot{\psi} \end{pmatrix}$$

Now introduce linear combinations

$$\phi = \frac{1}{2}\left(\psi + \frac{i}{m}\dot{\psi}\right) \qquad \chi = \frac{1}{2}\left(\psi - \frac{i}{m}\dot{\psi}\right)$$

"Foldy-Wouthuysen" trafo. In the non-relativistic limit, have

$$0 \leq E_{\vec{p}} - m \ll 2m$$

Searching theory that only contains d.o.f. that are relevant in the NR limit.

Now lets assume $\psi(x)$ is given by a plane wave:

$$\psi(x) = \exp(-i(E_{\vec{p}}t - \vec{p}\vec{x})) \qquad E_{\vec{p}} = +\sqrt{\vec{p}^2 + m^2}$$

Then $\chi(x)$ vanishes in the NR limit:

$$\chi(x) = \frac{1}{2}\left(1-\frac{E}{m}\right)\psi(x) = \frac{1}{2}\left(1-\sqrt{1+\frac{\vec{p}^2}{m^2}}\right)\psi(x)$$

$$= \mathcal{O}(v^2)\,\psi(x)$$

$\hookrightarrow$ small if $v \ll 1$, i.e. in NR limit

Rewrite ODE with ϕ, χ:

$$\frac{\partial}{\partial t}\begin{pmatrix}\phi \\ \chi\end{pmatrix} = \begin{pmatrix} \frac{i}{2m}\Delta - im & \frac{i}{2m}\Delta \\ -\frac{i}{2m}\Delta & -\frac{i}{2m}\Delta + im \end{pmatrix}\begin{pmatrix}\phi \\ \chi\end{pmatrix}$$

Setting χ to zero, we find

$$\frac{\partial}{\partial t}\phi(x) = \left(\frac{i}{2m}\Delta - im\right)\phi(x)$$

$$\Leftrightarrow \quad i\frac{\partial}{\partial t}\phi(x) = \underbrace{\left(-\frac{1}{2m}\Delta + m\right)}_{\hat{H} = -\frac{\Delta}{2m} + m}\phi(x)$$

4.1.4 Lagrangian for scalar field theory

Peskin, sec 2.2

Action S, is time integral of Lagrange function L. For local field theories, L can be written as spatial integral of a Lagrangian $\mathcal{L}$:

$$S = \int L\,dt = \int d^4x\, \mathcal{L}(\phi, \partial_\mu \phi)$$

i.e. $\mathcal{L}$ depends on ϕ and its derivatives.

Equation of motion obtained from variation of action (principle of least action):

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \, .$$

[compare: $\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \left(\frac{\partial L}{\partial q}\right) = 0$]

Example: $\mathcal{L}_{KG} = \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - \frac{1}{2} m^2 \phi^2$

EOM yield

$$\partial_\mu \partial^\mu \phi + m^2 \phi = 0$$

$$\Leftrightarrow (\Box + m^2)\phi = 0 \qquad \text{Klein Gordon eqn.}$$

Hamiltonian formulation:

Reminder: $p = \frac{\partial L}{\partial \dot{q}}$, $H = \sum p\dot{q} - L$ (q generalised coords)

Poisson brackets: $\{q_i, p_j\} = \delta_{ij}$

$$\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}$$

Quantization: $\{,\} \to [,]$

Define: ~~$p(\vec{x})\,\forall$~~

$$\pi(\vec{x}) = \frac{\partial \mathcal{L}}{\partial(\partial\dot{\phi}(\vec{x}))}$$ conjugate momentum to $\phi(\vec{x})$

Motivation: $$p(\vec{x}) = \frac{\partial L}{\partial \dot{\phi}(\vec{x})} = \frac{\partial}{\partial \dot{\phi}(\vec{x})} \int d^3y\, \mathcal{L}(\phi(\vec{y}), \dot{\phi}(\vec{y}))$$

$$\approx \frac{\partial}{\partial \dot{\phi}(\vec{x})} \sum_{\vec{y}} \mathcal{L}(\phi(\vec{y}), \dot{\phi}(\vec{y}))\, d^3y$$

finite volume elements

$$= \pi(\vec{x})\, d^3x$$

$\Rightarrow$ Hamiltonian

$$H = \int d^3x \left(\pi(\vec{x})\dot{\phi}(\vec{x}) - \mathcal{L} \right) \equiv \int d^3x\, \mathcal{H}$$

For $\mathcal{L}_{KG}$, find $\pi(\vec{x}) = \dot{\phi}(\vec{x})$

$$\Rightarrow \mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2$$

Noether's theorem

Connection between symmetries and conservation laws

If $\mathcal{L}$ is invariant (up to total derivatives) under a symmetry transformation

$$\phi(x) \to \phi'(x) = \phi(x) + \alpha \Delta\phi(x)$$

α ↳ small parameter

then the equ. of motions are unchanged → symmetry.

$$\Rightarrow \quad \mathcal{L}(x) \to \mathcal{L}(x) + \alpha\, \partial_\mu \mathcal{J}^\mu(x)$$

does not contribute to action locally, only on boundary.

Variation of $\mathcal{L}$:

$$\alpha \Delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi}(\alpha \Delta\phi) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu(\alpha \Delta \phi)$$

$$= \alpha\, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi \right) + \alpha \underbrace{\left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right)}_{\hookrightarrow 0} \Delta\phi$$

$$\Rightarrow \quad \partial_\mu j^\mu = 0 \quad \text{for} \quad j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi - \mathcal{J}^\mu$$

⇒ conserved current j^μ for each continuous symmetry

In particular, $Q = \int d^3x\, j^0$ is constant, if all fields vanish at ∞.

Examples: $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2$ invariant under $\phi \to \phi + \alpha$

$\Rightarrow j^\mu = \partial^\mu \phi$ conserved

$\mathcal{L} = |\partial_\mu \phi|^2 + m^2|\phi|^2$ $\phi \in \mathbb{C}$, invariant under $\phi \to e^{i\alpha}\phi$
$= \phi + i\alpha\phi$

$$\Rightarrow j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \Delta\phi^* = i\left((\partial^\mu \phi^*)\phi - \phi^*(\partial^\mu\phi)\right)$$

Finally, a spacetime trafo $x^\mu \to x^\mu - a^\mu$ can be described as a field transformation:

$$\phi(x) \to \phi(x+a) = \phi(x) + a^\mu \partial_\mu \phi(x)$$

In particular, $\mathcal{L}(x) \to \mathcal{L}(x+a) = \mathcal{L}(x) + a^\mu \partial_\mu \mathcal{L}(x)$

$$\Rightarrow (j^\mu)_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)} \partial_\nu \phi - \mathcal{L}\,\delta^\mu{}_\nu \equiv T^\mu{}_\nu$$

"energy momentum tensor"

Hamiltonian: $H = \int T^{00} d^3x \equiv \int \mathcal{H}\, d^3x$ $\Leftrightarrow$ time translations

Spatial momenta:

$$P^i = \int T^{0i} d^3x = -\int \pi\, \partial_i \phi\, d^3x$$

physical momenta, generate spatial translations

Inset: From discrete to continuum mechanics:
(Weinzierl EM script)

Infinitely long rod in 1-d, model as masses connected by springs

Lagrange fn: $\quad T = \frac{1}{2} \sum_{i \in \mathbb{Z}} m \dot{q}_i^2 \qquad V = \frac{1}{2} \sum_{i \in \mathbb{Z}} k (q_{i+1} - q_i)^2$

$$L = T - V = \frac{1}{2} \sum_{i \in \mathbb{Z}} \left(m \dot{q}^2 - k (q_{i+1} - q_i)^2 \right) = \frac{1}{2} \sum_{i \in \mathbb{Z}} a \left(\frac{m}{a} \dot{q}_i^2 - ka \left(\frac{q_{i+1} - q_i}{a} \right)^2 \right)$$

EOM: $\quad \frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = 0 \qquad \forall i \in \mathbb{Z}$

$$\Leftrightarrow \quad ka \left(\frac{q_{i+1} - 2q_i - q_{i-1}}{a^2} \right) - \frac{m}{a} \ddot{q}_i = 0$$

Now take $a \to 0$ limit, i.e. replace $i \in \mathbb{Z}$ with $x \in \mathbb{R}$.

$q_x(t) \to q(t, x)$ as field on $\mathbb{R}^2$

Replace $\quad \frac{m}{a} \to \mu$ (mass density (in 1D))

$\qquad ka \to Y$ (Young module), relation between force and stretch
$F(t,x) = Y \xi(t,x)$

and $\quad \xi(t,x) = \lim_{a \to 0} \frac{q_{i+1}(t) - q_i(t)}{a} = \lim_{a \to 0} \frac{q(t, x+a) - q(t,x)}{a} = \frac{\partial}{\partial x} q(t,x)$

Finally the sum becomes an integral, and

$$L = \frac{1}{2}\int_{-\infty}^{+\infty} dx \underbrace{\left[\mu\left(\frac{\partial q(t,x)}{\partial t}\right)^2 - Y\left(\frac{\partial q(t,x)}{\partial x}\right)^2\right]}_{\text{Lagrange density}}$$

$S = \int dt \int dx \, \mathcal{L}$. Variation $q(t,x) \to q(t,x) + \delta q(t,x)$, with $\delta q = 0$ at $x = \pm\infty$, $t = \pm\infty$.

$$\Rightarrow \delta S = \int dt\, dx \left(\frac{\partial \mathcal{L}}{\partial(\partial_t q)} \delta \partial_t q + \frac{\partial \mathcal{L}}{\partial(\partial_x q)} \delta \partial_x q \right)$$

$$= -\int dt\, dx \left[\partial_t \frac{\partial \mathcal{L}}{\partial(\partial_t q)} + \partial_x \frac{\partial \mathcal{L}}{\partial(\partial_x q)} \right] \delta q$$

General case, $\mathcal{L}(\psi(x), \partial_\mu \psi(x), j(x), x)$

Find that action is stationary if

$$\frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = 0$$

Quantization ? (PS, 2.3)

Promote fields to operators, impose commutation relations.

For discrete systems, used

$$[\hat{q}_i, \hat{p}_j] = i\delta_{ij} \quad ; \quad [\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0 .$$

Generalisation: $\left[\hat{\phi}(\vec{x}), \hat{\pi}(\vec{y})\right] = i\,\delta^3(\vec{x}-\vec{y})$

$$\left[\hat{\phi}(\vec{x}), \hat{\phi}(\vec{y})\right] = \left[\hat{\pi}(\vec{x}), \hat{\pi}(\vec{y})\right] = 0 .$$

(time dependence later) Note: Omit ^ from now on ! ! !

Interested in spectrum of $H \to \hat{H}$. Go to Fourier space:

$$\phi(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3}\, e^{i\vec{p}\vec{x}}\, \phi(\vec{p},t) \qquad \left[\phi^*(\vec{p}) = \phi(-\vec{p}) \text{ for } \phi(\vec{x}) \in \mathbb{R}\right]$$

KG eqn: $\left(\partial_t^2 - \nabla^2 + m^2\right)\phi(\vec{x},t) = 0$

$$\Rightarrow \quad \left(\partial_t^2 + |\vec{p}|^2 + m^2\right)\phi(\vec{p},t) = 0$$

$\Rightarrow$ Harmonic oscillator with frequency $\omega_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$

Solve using ladder operators:

$$\phi = \frac{1}{\sqrt{2\omega}}\left(a + a^+\right) \qquad p = -i\sqrt{\frac{\omega}{2}}\left(a - a^+\right)$$

with $[a, a^\dagger] = 1$, $\quad H_{\substack{\text{one} \\ \text{oscillator}}} \to \omega\left(a^\dagger a + \frac{1}{2}\right)$

ground state $|0\rangle$ with $a|0\rangle = 0$ and $H|0\rangle = \frac{\omega}{2}|0\rangle$.

$|n\rangle = (a^\dagger)^n |0\rangle$ with $H|n\rangle = \omega\left(n + \frac{1}{2}\right)|n\rangle$.

Do the same here, for ∞ many harmonic oscillators

$$\phi(\vec{x}, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_{\vec{p}}\, e^{i\vec{p}\vec{x}} + a^\dagger_{\vec{p}}\, e^{-i\vec{p}\vec{x}} \right)$$

$$\pi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_{\vec{p}}}{2}} \left(a_{\vec{p}}\, e^{i\vec{p}\vec{x}} - a^\dagger_{\vec{p}}\, e^{-i\vec{p}\vec{x}} \right)$$

and $[a_{\vec{p}}, a^\dagger_{\vec{p}'}] = (2\pi)^3 \delta^3(\vec{p} - \vec{p}')$

Now check that

$$[\phi(\vec{x}), \pi(\vec{x}')] = \int \frac{d^3p\, d^3p'}{(2\pi)^6} \frac{-i}{2} \sqrt{\frac{\omega_{p'}}{\omega_p}} \Big(\underbrace{[a^\dagger_{-\vec{p}}, a_{\vec{p}'}]}_{-(2\pi)^3\delta^3(-\vec{p}+\vec{p}')} - \underbrace{[a_{\vec{p}}, a^\dagger_{-\vec{p}'}]}_{(2\pi)^3\delta^3(\vec{p}+\vec{p}')} \Big) e^{i(\vec{p}\vec{x} + \vec{p}'\vec{x}')}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{+i}{2} 2 e^{i(\vec{p}\vec{x} - \vec{p}\vec{x}')} = i\delta^3(\vec{x} - \vec{x}')$$

Hamiltonian:

$$H = \int d^3x \int \frac{d^3p\, d^3p'}{(2\pi)^6}\, e^{i(\vec{p}+\vec{p}\,')\vec{x}} \Big(-\frac{\sqrt{\omega_{\vec{p}}\,\omega_{\vec{p}'}}}{4} (a_{\vec{p}} - a^\dagger_{-\vec{p}})(a_{\vec{p}'} - a^\dagger_{-\vec{p}'})$$

$$+ \frac{-\vec{p}\vec{p}\,' + m^2}{4\sqrt{\omega_{\vec{p}}\,\omega_{\vec{p}'}}} \left(a_{\vec{p}} + a^\dagger_{-\vec{p}}\right)\left(a_{\vec{p}'} + a^\dagger_{-\vec{p}'}\right)\Big)$$

$$= \int \frac{d^3p}{(2\pi)^3}\, \omega_{\vec{p}} \Big(a^\dagger_{\vec{p}} a_{\vec{p}} + \underbrace{\frac{1}{2}[a_{\vec{p}}, a^\dagger_{\vec{p}}]}_{\propto\, \delta^3(0),\ \text{infinite}} \Big)$$

Usual argument, zero-point energy unobservable ...

Now, can check that

$$[H, a^\dagger_{\vec{p}}] = \omega_{\vec{p}}\, a^\dagger_{\vec{p}} \qquad [H, a_{\vec{p}}] = -\omega_p\, a_{\vec{p}}$$

$\Rightarrow$ Starting from a ground state $|0\rangle$, the states $a^\dagger_{\vec{p}} a^\dagger_{\vec{q}} \dots |0\rangle$ are Eigenstates of H with energies $\omega_{\vec{p}} + \omega_{\vec{q}} + \dots$

Furthermore the momentum operator

$$\vec{P} = -\int d^3x\, \pi(\vec{x})\, \nabla\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\, \vec{p}\, a^\dagger_{\vec{p}} a_{\vec{p}}$$

$\Rightarrow$ $a^\dagger_{\vec{p}}|0\rangle$ is a state with momentum $\vec{p}$, energy $\omega_{\vec{p}}$.

$a^\dagger_{\vec{p}} a^\dagger_{\vec{q}} |0\rangle$ —"— $\vec{p}+\vec{q}$, $\omega_{\vec{p}} + \omega_{\vec{q}}$

Natural interpretation: $a^\dagger_{\vec{p}}$ creates particles with energy $\omega_{\vec{p}}$ and momentum $\vec{p}$.

$\hookrightarrow$ Relativistic QM is a many body theory

Also, no problem with negative energies. Starting from the GS $|0\rangle$ with E_0, excited states have $E > E_0$. Meaningful quantum theory.

Statistics: $a^\dagger_{\vec{p}}, a^\dagger_{\vec{q}}$ commute, therefore

$$a^\dagger_{\vec{p}} a^\dagger_{\vec{q}} |0\rangle = a^\dagger_{\vec{q}} a^\dagger_{\vec{p}} |0\rangle$$

$\Rightarrow$ Bosons.

Causality & Propagator (PS 2.4)

Heisenberg picture: $\phi(x) = e^{iHt} \phi(\vec{x}) e^{-iHt}$

Can use $i \frac{\partial}{\partial t} \mathcal{O} = [\mathcal{O}, H]$ to show that $\phi(x)$ (as field operator) satisfies

$$\frac{\partial^2}{\partial t^2} \phi = (\nabla^2 - m^2) \phi$$

One can explicitly work out $\phi(x)$ from $\phi(\vec{x})$, using first that

$$H a_{\vec{p}} = a_{\vec{p}} (H - E_{\vec{p}})$$

and

$$H^n a_{\vec{p}} = a_{\vec{p}} (H - E_{\vec{p}})^n$$

$$\Rightarrow \quad e^{iHt} a_{\vec{p}} e^{-iHt} = a_{\vec{p}} e^{i(H-E_{\vec{p}})t} e^{-iHt} = a_{\vec{p}} e^{-iE_{\vec{p}}t}$$

and $$e^{iHt} a^\dagger_{\vec{p}} e^{-iHt} = a^\dagger_{\vec{p}} e^{iE_{\vec{p}}t} .$$

It follows that $\phi(x) = e^{iHt} \phi(\vec{x}) e^{-iHt}$ is given by

$$\phi(x) = \phi(\vec{x}, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}} e^{-ipx} + a^\dagger_{\vec{p}} e^{ipx} \right) \Big|_{p_0 = E_{\vec{p}}} .$$

$$\pi(x) = \frac{\partial}{\partial t} \phi(x) .$$

Now, $\phi(x)$ contains both e^{-ip^0t} and e^{+ip^0t} terms. However now the interpretation is different: These are merely the coefficients of the operators that annihilate or create states with $(E_{\vec{p}}, \vec{p})$. The theory has a well defined ground state now.

Causality

The amplitude for a particle propagating from y to x is

$$\langle 0 | \phi(x) \phi(y) | 0 \rangle \equiv D(x-y)$$

Inserting $\phi(x)$, the only contributions come from terms

$$\langle 0 | a_{\vec{p}} a^\dagger_{\vec{q}} | 0 \rangle = (2\pi)^3 \delta^3(\vec{p} - \vec{q})$$

$$\Rightarrow D(x-y) = \underbrace{\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}}}_{\text{Note: This is a Lorentz invariant integral}} e^{-ip(x-y)}$$

To investigate causality, we evaluate it for different choices of $x-y$ ...

1. Timelike difference $x^0 - y^0 \equiv t$, $\vec{x} - \vec{y} = 0$.

$$D(x-y) = \frac{4\pi}{(2\pi)^3} \int_0^\infty dp \frac{p^2}{2\sqrt{p^2+m^2}} e^{-i\sqrt{p^2+m^2}\,t}$$

$$= \frac{1}{4\pi^2} \int_m^\infty dE \sqrt{E^2-m^2}\, e^{-iEt}$$

$$\sim e^{-imt} \quad \text{for } t \to \infty \quad (\text{Mathematica!})$$

Bessel fn,

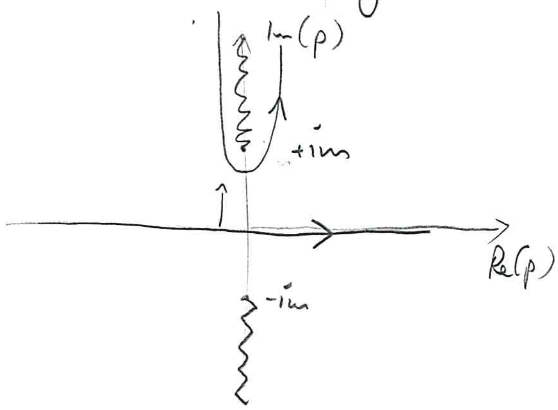
2. Spacelike distance $x - y = \begin{pmatrix} 0 \\ \vec{r} \end{pmatrix}$

$$D(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{i\vec{p}\vec{r}}$$

$$= \frac{2\pi}{(2\pi)^3} \int_0^\infty dp \frac{p^2}{2E_{\vec{p}}} \frac{e^{ipr} - e^{-ipr}}{ipr}$$

$$= \frac{-i}{2(2\pi)^2} \frac{1}{r} \int_{-\infty}^{+\infty} dp \, \underbrace{\frac{p e^{ipr}}{\sqrt{p^2+m^2}}}_{\text{complex fn of } p \text{, with branch cut starting at } \pm im}$$

Push contour around upper branch cut



With $\rho = -ip$, find

$$\frac{1}{4\pi^2 r} \int_m^\infty d\rho \, \frac{\rho}{\sqrt{\rho^2 - m^2}} \, e^{-m r}$$

$$\sim e^{-mr} \qquad (r \to \infty) \qquad \text{(Mathematica)}$$

Nonzero outside of light cone ↯ causality ?
Or are we asking the wrong question ?

Can a measurement at x influence one at y, if $(x-y)^2 < 0$?
If we measure ϕ, then we ask whether $[\phi(x), \phi(y)]$ is nonzero :

$$[\phi(x), \phi(y)] = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \times$$

$$\times \left[a_{\vec p} e^{-ipx} + a^\dagger_{\vec p} e^{ipx} , \; a_{\vec q} e^{-iqy} - a^\dagger_{\vec q} e^{iqy} \right]$$

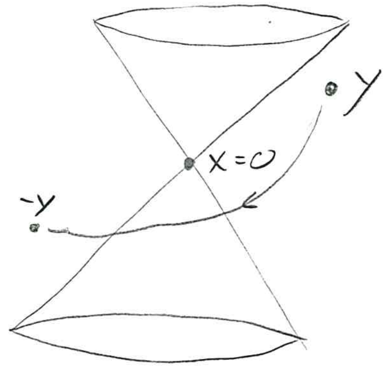
$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec p}} \left(e^{-ip(x-y)} - e^{ip(x-y)} \right)$$

$$= D(x-y) - D(y-x) .$$

Observation : $[\phi(x), \phi(y)]$, a product of operators, is just a number !?

If $(x-y)^2 < 0$, there exists a Lorentz trafo that takes $(x-y)$ to $-(x-y)$. Due to Lorentz invariance of D, it follows that $[\phi(x), \phi(y)] = 0$ for $(x-y)^2 < 0$.

If $(x-y)^2 > 0$, then no such LT exists, and $[\phi(x), \phi(y)]$ is non-vanishing.



Next: Klein Gordon propagator as Green's fn of free scalar

Particle creation by a classical source

Klein Gordon field coupled to source $j(x)$:

$$(\partial^2 + m^2)\,\phi(x) = j(x)$$

$\uparrow$ non-zero for finite time interval

Start in vacuum state before j ~~(x)~~ is turned on, can ask what the state of the system is after j is switched off

$$\phi_0(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_{\vec p}\, e^{-ipx} + a^\dagger_{\vec p}\, e^{ipx} \right)$$

Solution for ~~the~~ inhomogeneous system using Green's function = propagator:

$$\phi(x) = \phi_0(x) + i \int d^4y\, D_R(x-y)\, j(y)$$

$$= \phi_0(x) + i \int d^4y \int \frac{d^3p}{(2\pi)^3} \cancel{\frac{1}{\sqrt{2E_p}}} \frac{1}{2E_{\vec p}}\, \theta(x^0 - y^0)$$

$$\times \left(e^{-ip(x-y)} - e^{ip(x-y)} \right) j(y)$$

Take x^0 large enough, so that $j=0$ ~~for~~ for all times $> x^0$. Then $\theta(x^0-y^0)$ can be replaced by 1.

Then the y-integral gives the Fourier trafo of j:

$$\tilde{j}(p) = \int d^4y \; e^{ipy} j(y) \qquad \text{, evaluated at } p^2 = m^2 .$$

Combined with ϕ_0, find:

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left[\left(a_{\vec{p}} + \frac{i}{\sqrt{2E_p}} \tilde{j}(p) \right) e^{-ipx} + h.c. \right]$$

Inserted in the Hamiltonian, we find

$$H = \int \frac{d^3p}{(2\pi)^3} E_{\vec{p}} \left(a^\dagger_{\vec{p}} - \frac{i}{\sqrt{2E_{\vec{p}}}} \tilde{j}^*(p) \right) \left(a_{\vec{p}} + \frac{i}{\sqrt{2E_{\vec{p}}}} \tilde{j}(\vec{p}) \right)$$

and

$$\langle 0 | H | 0 \rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} |\tilde{j}(p)|^2$$

This is equivalent to a system with ~~probability~~ $\frac{|\tilde{j}(p)|^2}{2E_{\vec{p}}}$ ~~to find a particle~~ particles in the state $\vec{p}$. We can therefore interpret $\frac{|\tilde{j}(\vec{p})|^2}{2E_{\vec{p}}}$ as the probability to produce a particle in the state $\vec{p}$.

Total particle number after the source has acted:

$$\int dN = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} |\tilde{j}(p)|^2$$

Only the Fourier modes of $j(x)$ for which $p^2 = m^2$ are effective at producing particles. → Resonances

Dirac equation (4.2 Weinzierl)

Since $\phi(x)$ is a scalar, i.e. invariant under rotations, it can not describe a Spin $\frac{1}{2}$ particle. Want to find relativistic eqn for spin $\frac{1}{2}$ particles, which is covariant.

Dirac tried to find an equation which is first order in time derivatives. Ansatz:

$$i\frac{\partial}{\partial t}\psi(x) = \frac{1}{i}\left(\alpha_1 \partial_{x^1} + \alpha_2 \partial_{x^2} + \alpha_3 \partial_{x^3} + \beta m\right)\psi(x)$$

The RHS should be invariant under rotations $\Rightarrow$ The α_i have to transform apropriately, can not be just numbers.

If α, β are $N\times N$ matrices, then

$$\psi(x) = \begin{pmatrix}\psi_1\\ \vdots \\ \psi_N\end{pmatrix}$$ in an N-dim vector, of N scalar functions.

$\Rightarrow$ ψ_k should obey KG eqn in addition:

$$(\Box + m^2)\psi_k = 0 \quad \text{or} \quad -\partial_t^2 \psi_k = (-\nabla^2 + m^2)\psi_k$$

Also have

$$-\partial_t^2 \psi_k = (i\partial_t)^2 \psi_k = \left[\frac{1}{i}\left(\alpha_1 \partial_{x^1} + \alpha_2 \partial_{x^2} + \alpha_3 \partial_{x^3}\right) + \beta m\right]^2 \psi_k$$

$$= -\sum_{i,j=1}^{3} \frac{1}{2}\left(\alpha_i \alpha_j + \alpha_j \alpha_i\right)\frac{\partial^2 \psi_k}{\partial x^i \partial x^j} + \frac{m}{i}\sum_{i=1}^{3}\left(\alpha_i \beta + \beta \alpha_i\right)\frac{\partial \psi_k}{\partial x^i} + m^2 \beta^2 \psi_k$$

⇒ Constraints

$$\alpha_i \alpha_j + \alpha_j \alpha_i = 2\,\delta_{ij}\,\mathbb{1} \quad \leftarrow N\times N \text{ identity matrix} \qquad (\Rightarrow \alpha_i^2 = \mathbb{1})$$

$$\alpha_i \beta + \beta \alpha_i = 0$$

$$\beta^2 = \mathbb{1}$$

Furthermore, the RHS of the Dirac eqn is the Hamiltonian, and therefore should be hermitian

$$\Rightarrow \qquad \alpha_i^\dagger = \alpha_i \qquad \beta^\dagger = \beta$$

~~Neither~~ None of α_i, β can be proportional to $\mathbb{1}$, otherwise it would follow that some other matrix is zero, in conflict with

$$\alpha_i^2 = \beta^2 = \mathbb{1} .$$

This also tells us that α_i, β have ~~eig~~ eigenvalues ± 1.

Finally ~~they~~ the α_i are traceless:

$$\mathrm{Tr}(\alpha_i) = \mathrm{Tr}(\beta^2 \alpha_i) = \mathrm{Tr}(\beta\, \alpha_i \beta) = -\mathrm{Tr}(\alpha_i)$$

↳ using $\alpha_i \beta = -\beta \alpha_i$

A traceless matrix with eigenvalues ± 1 must be even dimensional

$$N = 2, 4, 6, 8, \ldots$$

Try $N=2$: Only 4 independent hermitian matrices, one of which is the identity $\Rightarrow$ impossible to satisfy anti-commutation relations
(Note: In 2 or 3 dimensions this could be satisfied)

$N=4$: One solution given by

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Last step: Multiply ~~Dira~~ eqn. with β, define

$$\gamma^0 \equiv \beta \quad ; \quad \gamma^i = \beta\,\alpha_i$$

and write $\gamma^\mu = (\gamma^0, \gamma^1, \gamma^2, \gamma^3)$ as 4-vector. Find Dirac equation:

$$\left(i\gamma^0\partial_0 + \gamma^1\partial_1 + \gamma^2\partial_2 + \gamma^3\partial_3 - m \right) \psi(x) = 0$$

or $$(i\gamma^\mu\partial_\mu - m)\,|\psi\rangle = 0 .$$

Introduce Dirac slash notation: $\gamma^\mu a_\mu \equiv \not{a}$

$$\Rightarrow \quad (i\not{\partial} - m)\,\psi = 0$$

Properties of γ^μ :

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\,\mathbb{1} \quad \text{as } 4\times 4 \text{ matrices}$$

Clifford - Algebra

Hermeticity:

$$(\gamma^0)^\dagger = \beta^\dagger = \beta = \gamma^0 \quad \text{(hermitian)}$$

$$(\gamma^i)^\dagger = (\beta\alpha_i)^\dagger = \alpha_i\beta = -\beta\alpha_i = -\gamma^i \quad \text{(anti-herm.)}$$

Furthermore:

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{24}\,\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$$

Can show that

$$\{\gamma^\mu, \gamma_5\} = 0 .$$

With our choice of α_i, β, we have

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \qquad \gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

This is known as the Dirac representation. Not unique.

Other examples: Weyl representation

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} \qquad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

useful for massless particles.

Projection matrices

$$P_{L,R} = \frac{1}{2}(1 \mp \gamma_5)$$

$$P_L = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad P_R = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$ project on left/right

handed solutions of Dirac eqn. For massless particles, they evolve independently:

Write $\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}$

in Weyl representation.

$$i\gamma^0 \partial_0 \psi = (i\gamma_i \partial_i + m)\psi$$

$$\Rightarrow \quad i\partial_0 \psi_R = i\sigma_i \partial_i \psi_R + m\psi_L$$

$$i\partial_0 \psi_L = i\sigma_i \partial_i \psi_L + m\psi_R$$

4-current for spin $\frac{1}{2}$ particles (4.2.1)

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

The hermitian conjugate is $\psi^+ = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*)$

Define:

$$j^\mu = (\rho, \vec{j})$$

with $\rho = \psi^+ \psi$, $\vec{j} = \psi^+ \vec{\alpha} \psi$ $\qquad \vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$

This satisfies a continuity equation:

$$\frac{\partial}{\partial t} \rho + \vec{\nabla} \vec{j} = 0$$

Proof: $\frac{\partial}{\partial t}(\psi^+ \psi) = \psi^+ \left(\frac{\partial}{\partial t} \psi \right) + \left(\frac{\partial}{\partial t} \psi^+ \right) \psi$

$$\psi^+ \frac{\partial}{\partial t} \psi = \psi^+ (-\alpha_i \partial_i - i\beta m) \psi \qquad (1)$$

$$\left(\frac{\partial}{\partial t} \psi \right)^+ \psi = \left[(-\alpha_i \partial_i - i\beta m) \psi \right]^+ \psi$$

$$= \psi^+ (-\alpha_i \overleftarrow{\partial}_i + i\beta m) \psi \qquad (2)$$

$$(1) + (2) = -\partial_i (\psi^+ \alpha_i \psi) = \frac{\partial}{\partial t} \rho .$$

In terms of γ^μ :

$$\rho = \psi^\dagger \psi = \psi^\dagger \gamma^0 \gamma^0 \psi = (\psi^\dagger \gamma^0) \gamma^0 \psi$$

$$j^i = \psi^\dagger \alpha_i \psi = \psi^\dagger \gamma^0 \gamma^0 \alpha_i \psi = (\psi^\dagger \gamma^0) \gamma^i \psi$$

$\Rightarrow$ Introduce adjoint spinor

$$\boxed{\bar\psi = \psi^\dagger \gamma^0} \qquad \Rightarrow \quad j^\mu = \bar\psi \gamma^\mu \psi$$

with $\partial_\mu j^\mu = 0$.

For $\bar\psi$, have

$$\bar\psi(-i\gamma^\mu \overleftarrow{\partial}_\mu - m) = 0$$

$\rightarrow$ Exercise.
Use $\gamma^0 (\gamma^\mu)^\dagger \gamma^0 = \gamma^\mu$

4.2.2 Solutions of the Dirac eqn

$$(i\not\partial - m)\psi = 0$$

Expect 4 solutions, since ψ has 4 components.

Two positive energy solutions, one for each spin $\uparrow, \downarrow$ particle

Two negative energy solutions, for the corresponding anti-particles.

Let $\epsilon_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$ and $p^\mu = (\epsilon_{\vec{p}}, \vec{p})$

Ansatz for positive energy solutions:

$$\psi(x) = e^{-ipx} u(p, \lambda)$$

spin quantum #.

$$\Rightarrow \quad (\not{p} - m) u(p, \lambda) = 0 .$$

Furthermore $\bar{\psi}(x) = \bar{u}(p,\lambda) e^{ipx}$ with $\bar{u}(p,\lambda)(\not{p} - m) = 0$

For negative energy solutions

$$\psi(x) = e^{ipx} v(p,\lambda) \; : \quad (\not{p} + m) v(p,\lambda) = 0$$

and analogeous $\bar{v}(p,\lambda)(\not{p} + m) = 0$.

Finally the solutions should be orthogonal

$$\Rightarrow \quad \bar{u}(p,\bar{\lambda}) u(p,\lambda) = 2m \, \delta_{\bar{\lambda}\lambda}$$

$$\bar{v}(p,\bar{\lambda}) v(p,\lambda) = 2m \, \delta_{\bar{\lambda}\lambda}$$

$$\bar{u}(p,\bar{\lambda}) v(p,\lambda) = \bar{v}(\bar{\lambda}) u(\lambda) = 0 .$$

The normalisation is arbitrary, but convenient

Massless case:

· Note: Also useful in highly relativistic limit, e.g. LHC proton mass ~ 1 GeV
beam energy 14 000 GeV
= 14 TeV.

Eqns reduce to $\not{p}\, u(p,\lambda) = 0 \qquad \bar{u}(p,\lambda)\, \not{p} = 0$

$$\not{p}\, v(p,\lambda) = 0 \qquad \bar{v}(p,\lambda)\, \not{p} = 0$$

$\Rightarrow$ sufficient to solve eqn's for $u, \bar{u}$. Choose Weyl representation:

$$\not{p} = \begin{pmatrix} 0 & 0 & p^0+p^3 & p^1-ip^2 \\ 0 & 0 & p^1+ip^2 & p^0-p^3 \\ p^0-p^3 & -p^1-ip^2 & 0 & 0 \\ -p^1+ip^2 & p^0+p^3 & 0 & 0 \end{pmatrix}$$

A few more definitions:

- light cone coordinates:

$$p^+ = \frac{1}{\sqrt{2}}(p^0+p^3) \qquad p^- = \frac{1}{\sqrt{2}}(p^0-p^3)$$

$$p^\perp = \frac{1}{\sqrt{2}}(p^1+ip^2) \text{ and } (p^\perp)^* = p^{\perp *}$$

For massless particles: $p_\mu p^\mu = 0 \iff p_+ p_- = p^\perp p^{\perp *}$

- σ^μ - matrices:

$$\sigma^\mu = (1, -\vec{\sigma}) \qquad \bar{\sigma}^\mu = (1, \vec{\sigma}) \qquad \vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$$

Then: $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$ and $\not{p} = \begin{pmatrix} 0 & p_\mu \sigma^\mu \\ p_\mu \bar{\sigma}^\mu & 0 \end{pmatrix}$

Furthermore: $p_\mu \sigma^\mu = \sqrt{2} \begin{pmatrix} p^+ & p^{\perp *} \\ p^\perp & p^- \end{pmatrix} \qquad p_\mu \bar{\sigma}^\mu = \sqrt{2} \begin{pmatrix} p^- & -p^{\perp *} \\ -p^\perp & p^+ \end{pmatrix}$

Finally introduce 2-component spinors $|p+\rangle$, $|p-\rangle$ such that

$$u(p,+) = \begin{pmatrix} |p+\rangle \\ 0 \end{pmatrix} \qquad u(p,-) = \begin{pmatrix} 0 \\ |p-\rangle \end{pmatrix} .$$

$|p+\rangle$ and $|p-\rangle$ are often called Weyl spinors.

With this, the Dirac equations reduce to

$$p_\mu \bar{\sigma}^\mu |p+\rangle = 0 \qquad p_\mu \sigma^\mu |p-\rangle = 0$$

$$\langle p+| p_\mu \bar{\sigma}^\mu = 0 \qquad \langle p-| p_\mu \sigma^\mu = 0$$

In components:

$$\begin{pmatrix} p^- & -p^{\perp *} \\ -p^\perp & p^+ \end{pmatrix} |p+\rangle = 0 \qquad \begin{pmatrix} p^+ & p^{\perp *} \\ p^\perp & p^- \end{pmatrix} = 0$$

This is a simple linear algebra problem, for eigenvectors with eigenvalue 0. Find

$$|p+\rangle = c_1 \begin{pmatrix} p^{\perp *} \\ p^- \end{pmatrix} \qquad |p-\rangle = c_2 \begin{pmatrix} p^- \\ -p^{\perp} \end{pmatrix}$$

$$\langle p+| = c_3 (p^{\perp}, p^-) \qquad \langle p-| = c_4 (p^-, -p^{\perp *})$$

[Furthermore $\bar{u}(p,+) = u^+(p,+)\gamma^0$ etc]

Normalisation: $\langle p\pm|\gamma^\mu|p\pm\rangle = 2p^\mu$

$[= \bar{u}(p\pm)\gamma^\mu u(p\pm)]$

$\Rightarrow \quad c_1 c_3 = \frac{\sqrt{2}}{p^-}$ and $c_2 c_4 = \frac{\sqrt{2}}{p^-}$.

Fixed up to little group scaling $\quad |p\pm\rangle \to \lambda |p\pm\rangle$

$\langle p\pm| \to \frac{1}{\lambda}\langle p\pm|$

$$|p+\rangle = \frac{2^{1/4}}{\sqrt{p^-}} \begin{pmatrix} p^{\perp *} \\ p^- \end{pmatrix}$$ etc, up to rescalings.

[Note: spinor products:

$$\langle p\, q\rangle = \langle p-|q+\rangle = \frac{\sqrt{2}}{\sqrt{p^-}\sqrt{q^-}} \left(p^- q^{\perp *} - q^- p^{\perp *}\right)$$

$$[q\, p] \equiv \langle q+|p-\rangle = \ldots$$

with: $\langle pq\rangle[qp] = 2p\cdot q$]

Massive case:

Let p be a four vector with $p^2 = m^2$, and q lightlike ($q^2=0$).

Then $$p^\flat = p - \frac{p^2}{2pq}\, q$$

is lightlike:

$$(p^\flat)^2 = p^2 - \frac{p^2}{pq}\, pq + \frac{p^2}{2pq}\, q^2 = 0 .$$

We define:

$$u(p,+) = \frac{1}{\langle p^\flat + | q - \rangle} (\not{p} + m) | q - \rangle$$
$$u(p,-) = \frac{1}{\langle p^\flat - | q + \rangle} (\not{p} + m) | q + \rangle$$

etc. Note, here we identify $|k+\rangle = \begin{pmatrix} |k+\rangle \\ 0 \end{pmatrix}$; $|k-\rangle = \begin{pmatrix} 0 \\ |k-\rangle \end{pmatrix}$.

Now $$(\not{p} - m)\, u(p,+) = \frac{1}{\langle p^\flat + | q - \rangle} (\not{p}^2 - m^2) | q - \rangle = 0$$

Calculation: $\not{p}^2 = p_\mu \gamma^\mu p_\nu \gamma^\nu = \frac{1}{2} p_\mu p_\nu \{\gamma^\mu, \gamma^\nu\} = p^2$

Can also check normalisation, orthogonality etc.

Note: Easier way. For massive spinor, solve in rest frame $p = \begin{pmatrix} m \\ \vec{0} \end{pmatrix}$. Then (in Dirac repr)

$$(\not p - m)\psi = \begin{pmatrix} (m-m)\mathbb{1} & 0 \\ 0 & 2m\,\mathbb{1} \end{pmatrix}\psi = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix}\psi \overset{!}{=} 0$$

$\Rightarrow$ Two independent solutions, e.g. $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$.

(And for anti-particles $p^0 = -m$, $\rightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$).

Then boost to arbitrary p. For that we need the Lorentz transformation properties of spinors:

Lorentz covariance of Dirac eqn.

(Schwabl 6.2
Weinzierl 4.2.3)

Principle of relativity: Laws of nature are the same in all inertial frames

equations of motion.

Inertial frames are related by Lorentz transformations

$$x' = \Lambda x \quad \Leftrightarrow \quad x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$$

with $\Lambda^{\lambda}{}_{\mu} g^{\mu\nu} \Lambda^{\rho}{}_{\nu} = g^{\lambda\rho}$.

Let the wave function in frame I' be ψ'. It must be possible to obtain it from ψ (in I).

$$\Rightarrow \quad \psi'(x') = F(\psi(x)) = F(\psi(\Lambda^{-1}x'))$$

The transformed field ψ' should satisfy the Dirac eqn in the frame I':

$$(i \gamma^\mu \partial'_\mu - m)\, \psi'(x') = 0 .$$

Since x and x' are related by a linear transformation, the same should be true for ψ, ψ':

$$\psi'(x') = S(\Lambda)\, \psi(x) = S(\Lambda)\, \psi(\Lambda^{-1} x') ,$$

where $S(\Lambda)$ is a 4×4 matrix in spinor space. Want to find $S(\Lambda)$, a representation of Λ on spinors.

Start with $(i\gamma^\mu \partial_\mu - m)\,\psi(x) = 0$

use: $\circ$ $\dfrac{\partial}{\partial x^\mu} = \dfrac{\partial x'^\nu}{\partial x^\mu} \dfrac{\partial}{\partial x'^\nu} = \Lambda^\nu{}_\mu\, \partial'_\nu$

$\circ$ $S^{-1} \psi'(x') = \psi(x)$

$$\Rightarrow \quad (i\gamma^\mu \Lambda^\nu{}_\mu\, \partial'_\nu - m)\, S^{-1}(\Lambda)\, \psi'(x') = 0 .$$

$$\Leftrightarrow \quad i S \Lambda^\nu{}_\mu\, \gamma^\mu S^{-1} \partial'_\nu\, \psi'(x') + m\, \psi'(x') = 0$$

Comparing with $(i\gamma^\nu \partial'_\nu - m)\,\psi'(x') = 0$ we find that Lorentz invariance holds if

$$S^{-1}(\Lambda)\, \gamma^\nu S(\Lambda) = \Lambda^\nu{}_\mu\, \gamma^\mu$$

Now consider infinitesimal Lorentz trafos

$$\Lambda^\nu{}_\mu = \delta^\nu{}_\mu + \Delta\omega^\nu{}_\mu$$

with $\Delta\omega^{\nu\mu}$ infinitesimal and anti-symmetric, $\Delta\omega^{\mu\nu} = -\Delta\omega^{\nu\mu}$.

Then $\Lambda^\lambda{}_\mu g^{\mu\nu} \Lambda^\rho{}_\nu = g^{\lambda\rho}$

becomes $(\delta^\lambda{}_\mu + \Delta\omega^\lambda{}_\mu) g^{\mu\nu} (\delta^\rho{}_\nu + \Delta\omega^\rho{}_\nu) = g^{\lambda\rho}$

$$\Leftrightarrow \quad g^{\lambda\rho} + \Delta\omega^\lambda{}_\mu g^{\mu\rho} + g^{\lambda\nu}\Delta\omega^\rho{}_\nu + \mathcal{O}(\Delta\omega^2) = g^{\lambda\rho}$$

$$\Leftrightarrow \quad g^{\lambda\rho} + \underbrace{\Delta\omega^{\lambda\rho} + \Delta\omega^{\rho\lambda}}_{=0} = g^{\lambda\rho} + \mathcal{O}(\Delta\omega^2).$$

$\Rightarrow$ $\Delta\omega$ has six independent, non-zero components. 3 rotations, 3 boosts. Consider one of each:

- $\Delta\omega^0{}_1 = -\Delta\omega^{01} = -\Delta\beta$ boost to a system moving with velocity $\Delta\beta$ in x direction

- $\Delta\omega^1{}_2 = -\Delta\omega^{12} = \Delta\varphi$ ~~boost~~ rotation to a system with angle $\Delta\varphi$ around z axis.

We expand S as series in $\Delta\omega$: lets write

$$S = \mathbb{1} + \tau, \quad S^{-1} = \mathbb{1} - \tau, \quad \text{with } \tau \ \mathcal{O}(\Delta\omega^{\mu\nu})$$

~~Then~~

Then: $(1-\tau)\gamma^\mu(1+\tau) = \gamma^\mu + \gamma^\mu\tau - \tau\gamma^\mu = \gamma^\mu + \Delta\omega^\mu{}_\nu\gamma^\nu$

$$\Rightarrow \quad \gamma^\mu\tau - \tau\gamma^\mu = \Delta\omega^\mu{}_\nu\gamma^\nu .$$

Finally with $\det(S) = 1$ we have $\det(S) = \det(1+\tau) = \det(1) + \mathrm{tr}(\tau)$
$\Rightarrow \mathrm{tr}(\tau) = 0$.

A solution is given by

$$\tau = \frac{1}{8}\Delta\omega^{\mu\nu}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) \equiv -\frac{i}{4}\Delta\omega^{\mu\nu}\sigma_{\mu\nu}$$

with $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$.

Rotation around z-axis:

$$\tau(R_3) = \frac{i}{2}\Delta\varphi\,\sigma_{12}$$

$$\sigma_{12} = \frac{i}{2}[\gamma_1, \gamma_2] = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} \qquad \text{(Dirac basis)}$$

$$S = 1 + \frac{i}{2}\Delta\varphi\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}.$$

A finite rotation ϑ is obtained by combining N infinitesimal rotations by $\frac{\vartheta}{N}$:

$$\psi'(x') = S(\vartheta)\psi(x) = \lim_{N\to\infty}\left(1 + \frac{i}{2N}\vartheta\sigma_{12}\right)^N \psi(x)$$

$$= e^{\frac{i}{2}\vartheta\sigma_{12}}\psi(x)$$

$$= \left(\cos\frac{\vartheta}{2} + i\sigma_{12}\sin\left(\frac{\vartheta}{2}\right)\right)\psi(x)$$

Note: For coordinate vectors

$$x' = \lim_{N\to\infty}\left(1+\frac{\vartheta}{N}\begin{pmatrix}0 & & & \\ & 0 & 1 & \\ & -1 & 0 & \\ & & & 0\end{pmatrix}\right)^N x = \begin{pmatrix}1 & & & \\ & \cos\vartheta & \sin\vartheta & \\ & -\sin\vartheta & \cos\vartheta & \\ & & & 1\end{pmatrix} x .$$

A rotation by $\vartheta = 2\pi$ returns x back to itself. But $S(2\pi) = -\mathbb{1}$, $S(4\pi) = \mathbb{1}$.
Spinors describe spin $\frac{1}{2}$ particles !

For boosts, we have $\Delta\omega^{01} = \Delta\beta$

$$\Sigma(L_1) = \frac{1}{2}\Delta\beta\,\gamma_0\gamma_1 = \frac{1}{2}\Delta\beta\,\alpha_1 .$$

~~Wit~~ For a finite velocity v, define $\tanh\eta = v$.

Have $L_1(\eta)x' = \begin{pmatrix}\cosh\eta & -\sinh\eta & & \\ -\sinh\eta & \cosh\eta & & \\ & & 0 & \\ & & & 0\end{pmatrix} x$

For the spinor, have

$$S(L_1) = \lim_{N\to\infty}\left(1+\frac{1}{2}\frac{\eta}{N}\alpha_1\right)^N = e^{\eta/2\,\alpha_1}$$

$$= \mathbb{1}\cosh\left(\frac{\eta}{2}\right) + \alpha_1\sinh\left(\frac{\eta}{2}\right)$$

Back to solving the Dirac eqn (Schwabl, Peskin)

For massive particles at rest, have

$$\psi^{(+)}(x) = u_r(m,0)\, e^{-imt},$$
$$\psi^{(-)}(x) = v_r(m,0)\, e^{imt}$$

And:
$$u_1(m,0) = \sqrt{m}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad u_2(m,0) = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$v_1(m,0) = \sqrt{m}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \qquad v_2(m,0) = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

These are eigenvectors of the Dirac Hamiltonian with Energies $\pm m$. Furthermore, they have eigenvalues $+1$ (for $r=1$) and -1 (for $r=2$) with respect to

$$\sigma^{12} = \begin{pmatrix}\sigma^3 & 0\\ 0 & \sigma^3\end{pmatrix}$$, the generator of rotations around the z-axis.

$\Rightarrow$ $r=1$ ($r=2$) corresponds to spin up (down).

For arbitrary p, the spinors satisfy

$$(\not{p}-m)\, u_r(p) = 0 \qquad (\not{p}+m)\, v_r(p) = 0$$

Furthermore $\not{p}\not{p} = p_\mu\gamma^\mu p_\nu\gamma^\nu = \frac{1}{2} p_\mu p_\nu \{\gamma^\mu,\gamma^\nu\} = p^2$

$$\Rightarrow (\not{p}-m)(\not{p}+m) = p^2 - m^2 = 0$$

Therefore any vector u multiplied with $(\not p+m)$ and any vector v multiplied with $(\not p-m)$ will be a solution:

$$(\not p-m)(\not p+m)\,u_r(m,0) = 0 \qquad \text{etc.}$$

Normalisation:

$$u_r(p) = \frac{\not p+m}{\sqrt{2m(m+E)}}\,u_r(m,0) = \begin{pmatrix} \sqrt{\frac{E+m}{2}}\,\chi_r \\ \frac{\vec\sigma\cdot\vec p}{(2(m+E))^{1/2}}\,\chi_r \end{pmatrix}$$

with $\chi_1 = \begin{pmatrix}1\\0\end{pmatrix}$ and $\chi_2 = \begin{pmatrix}0\\1\end{pmatrix}$.

Relativistic limit: $u_r(p) \to \frac{\not p}{\sqrt{2E}}\,\frac{u_r(m,0)}{\sqrt m} = \begin{pmatrix} \sqrt{E/2}\,\chi_r \\ \frac{\vec\sigma\cdot\vec p}{\sqrt{2E}}\,\chi_r \end{pmatrix}$.

Here it would be nicer to work in the Weyl basis...

Quantisation, spin statistics (Peskin 3.5)

Lagrangian of Dirac field.

$$\mathcal{L} = \bar\psi(i\not\partial-m)\psi$$

Check: $\frac{\partial\mathcal{L}}{\partial\bar\psi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar\psi)} = (i\not\partial-m)\psi = 0$.

Canonical conjugate momentum: $\frac{\partial\mathcal{L}}{\partial(\partial_0\psi)} = \bar\psi i\gamma^0 = i\psi^+$

Hamiltonian: (Reminder: $\int d^3x \; \Pi \dot{\varphi} - \mathcal{L}$)

$$H = \int d^3x \left(i\psi^+ \partial_0 \psi - \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \right)$$

$$= \int d^3x \left(i\bar{\psi}\gamma^0 \partial_0 \psi - \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \right)$$

$$= \int d^3x \left(\bar{\psi}(-i\vec{\gamma}\vec{\nabla} + m)\psi \right).$$

How not to quantise the Dirac field

Lets try to do the same as for the Klein Gordon field

(Reminder: $[\phi(\vec{x}), \Pi(\vec{y})] = i\delta^3(\vec{x}-\vec{y})$)

Postulate: $[\psi_a(\vec{x}), \psi_b^+(\vec{y})] = \delta^3(\vec{x}-\vec{y})\,\delta_{ab}$

Now, expand ψ, ψ^+ in terms of creation and annihilation operators that diagonalise H.

Remember: Solutions with $p^0 > 0$: $u_r(p)\, e^{-ipx}$

These are eigenfunctions of H with energy $p^0 = \sqrt{\vec{p}^2 + m^2}$;

$$(-i\vec{\gamma}\vec{\nabla} + m)\, u_r(p)\, e^{-ipx} = (\vec{\gamma}\vec{p} + m)\, u_r(p)\, e^{-ipx}$$

$$= \gamma_0 p^0 \, u_r(p)\, e^{-ipx} \qquad \text{using } (\not{p} - m)\, u_r(p) = 0 .$$

Similarly, $v_r(p)$ are eigenfunctions with $p^0 = -\sqrt{\vec{p}^2 + m^2}$.

$\Rightarrow$ expand ψ as

$$\psi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} e^{i\vec{p}\vec{x}} \sum_{s=1,2} \left(a^s_{\vec{p}} u_s(\vec{p}) + b^s_{-\vec{p}} v_s(-\vec{p}) \right)$$

With commutationrelations

$$[a^r_{\vec{p}}, a^{s\dagger}_{\vec{q}}] = [b^r_{\vec{p}}, b^{s\dagger}_{\vec{q}}] = (2\pi)^3 \delta^3(\vec{p}-\vec{q}) \delta^{rs}$$

Can check that this gives $[\psi(\vec{x}), \psi^\dagger(\vec{y})] = \delta^3(\vec{x}-\vec{y}) \cdot \mathbb{1}$

Now we can compute the Hamiltonian: (Note: Peskin says it is a short calculation ... probably several pages)

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_s \left(E_{\vec{p}} a^{s\dagger}_{\vec{p}} a^s_{\vec{p}} - E_{\vec{p}} b^{s\dagger}_{\vec{p}} b^s_{\vec{p}} \right)$$

Problem! No ground state (lowest energy state).

... extended discussion in Peskin $\rightarrow$ also causality broken

How to quantise the Dirac field

A consistent quantisation is possible by imposing anti-commutation relations between ψ and $\psi^\dagger$:

Expand the field as

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_s \left(a^s_{\vec{p}}\, u_s(p)\, e^{-ipx} + b^{s+}_{\vec{p}}\, v_s(p)\, e^{ipx} \right)$$

$$\bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(b^s_{\vec{p}}\, \bar{v}_s(p)\, e^{-ipx} + a^{s+}_{\vec{p}}\, \bar{u}_s(p)\, e^{ipx} \right)$$

With $\{a^r_{\vec{p}}, a^{s+}_{\vec{q}}\} = \{b^r_{\vec{p}}, b^{s+}_{\vec{q}}\} = (2\pi)^3 \delta^3(\vec{p}-\vec{q})\, \delta^{rs}$

$\{a,a\}=0 \quad \{a,b\}=0 \ ; \ \{a,b^+\}=0, \ \{a^+,b\}=0$ etc.

Can insert expansion and show that this gives

$$\{\psi_a(\vec{x}), \psi^+_b(\vec{y})\} = \delta^3(\vec{x}-\vec{y})\, \delta_{ab}$$

$$\{\psi_a, \psi_b\} = \{\psi^+_a, \psi^+_b\} = 0 .$$

Define the vacuum or ground state $|0\rangle$ via

$$a^s_{\vec{p}}|0\rangle = 0 \quad \text{and} \quad b^s_{\vec{p}}|0\rangle = 0 \qquad \forall \vec{p}, s .$$

and one finds for the Hamiltonian:

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_s E_{\vec{p}} \left(a^{s+}_{\vec{p}} a^s_{\vec{p}} + b^{s+}_{\vec{p}} b^s_{\vec{p}} \right) \quad (+\infty \ldots) .$$

Note: The spin statistics theorem:

For Lorentz-invariant, causal quantum field theories, which have positive energy for all particle states, and positive norm for all states, fields of integer spin commute, and fields of half-integer spin anti-commute.

Remark: Before, this was something we imposed on the fermions because we already knew e.g. the Pauli principle. This is more fundamental!

The end ... sort of

Path integrals in Quantum mechanics