Friederike Schmid, Institute of Physics SS 2019

# Exercises in "Advanced Statistical Physics"

**Problem set 1**, due May 22nd, 2019

## Problem 1) Ising Ladder

Consider an Ising model on a ladder geometry with periodic boundary conditions along the ladder. The degrees of freedom are Ising spins $S_{ij} = \pm 1$ with $i = 1, ..., N$ and $j = 1, 2$, and the energy function is given by

$$\mathcal{H} = -J \sum_{i=1}^{N} \Big( S_{i,1} S_{i,2} + S_{i,1} S_{i+1,1} + S_{i,2} S_{i+1,2} \Big)$$

where $S_{N+1,1} = S_{1,1}$ and $S_{N+1,2} = S_{1,2}$.

Calculate the free energy of this system using the transfer matrix method.

(a) First consider the canonical partition function and bring it in a transfer matrix form. Your vectors will be four dimensional and the transfer matrix a $4 \times 4$ matrix.

(b) Calculate the Eigenvalues of your transfer matrix. Use mathematica or another program for symbolic algebra. If mathematica does not find explicit expressions for the Eigenvalues, you have made a mistake.

Hint: Mathematica sometimes fails to find simple solutions if the expressions that are being fed in are too complicated. Simplify your matrix by writing it in terms of suitable variables, e.g., replace $\exp(\beta J)$ by $t$ or the like.

(c) Calculate the free energy and the specific heat in the limit $N \to \infty$.

## Problem 2) Correlation functions in the Ising chain

To calculate the correlation functions in the Ising chain, it is easiest to define a generalized Ising model where the interaction parameter $J$ varies from bond to bond. Consider an Ising model with the energy function

$$\mathcal{H} = -\sum_{i=1}^{N} J_i S_i S_{i+1}$$

Note that we do *not* impose periodic boundary conditions here, i.e., the first and the last spin have only one neighbour.

(a) In the absence of an external field $H$, it is easier to calculate the partition function recursively. Verify the recursion relation $Z_{N+1} = Z_N 2\cosh(\beta J_N)$. Use this to calculate the partition function

(b) Use the result of (a) to determine the correlation functions $G_{i,n} = \langle S_i S_{i+n} \rangle$. To this end, take suitable derivatives of $Z$ with respect to the parameters $J_k$ (taking into account $S_j^2 = 1$). The solution is $G_{i,n} = \prod_{k=i}^{n+i}(\tanh(\beta J_k))$.

(c) Consider now the regular Ising model, where all $J_k \equiv J$ are identical.
Show $G_{i,n} = e^{-n/\xi}$ and determine $\xi$.
Show how $\xi$ diverges as $T \to 0$ (i.e., $\beta = 1/k_B T \to \infty$).

**Problem 3) Ising model: Mean field solution**

The mean field solution of the Ising model is constructed in analogy to the Curie-Weiss approximation for ferromagnets.

Consider a general Ising model in $d$ dimensions on a regular $d$-cubic lattice (i.e., square in 2 dimensions, cubic in 3 dimensions etc.)

$$\mathcal{H} = -\sum_{\langle ij \rangle} J_i S_i S_j - H \sum_i S_i$$

Here the sum $\langle ij \rangle$ runs over nearest neighbor, and $i$ encodes a $d$-dimensional index $(n_x, n_y, ...)$ To construct the mean field approximation, we proceed as follows:

(a) First we consider a system of a *single* spin $S$ in an external field $H_{\text{eff}}$. Calculate the partition function an the statistical average $\langle S \rangle =: m$.

(b) From the point of view of an individual spin $S_i$, the interaction with the surrounding spins $S_j$ has the same effect as an effective field. Calculate the total effective field $H_{i,\text{eff}}[\{S_j\}] = -\partial\mathcal{H}/\partial S_i$. It still depends on the current state of the spins $S_j$.
Next, replace $S_j$ in the expression for $H_{i,\text{eff}}$ by their average value, $S_j \to \langle S_j \rangle = m$. Combining this with the result of (a), you will obtain an implicit equation for $m$.
The result is

$$m = \tanh(\beta(Jqm + H)) \tag{1}$$

(where you still have to determine the value of $q$).

(c) Starting from Eq. (1), consider the case $H = 0$. Unfortunately, it is not possible to give an explicit expression for $m(\beta)$. However, you can find a solution graphically by plotting $m$ and $f(m)$ versus $m$ and determining the crossing points.
Answer the following questions:
  - Plot $m$ and $f(m)$ for $\beta Jq = 0.5$ and $\beta Jq = 2$. What do you observe?
  - Determine the value of $\beta_c Jq$ where you can have more than one crossing point. This defines the critical point $\beta_c = 1/k_B T_c$.

(d) Make a Taylor expansion of $f(m)$ up to order $m^3$ and use this to show $m(T) \propto \sqrt{1 - T/T_c}$.