Exercises in "Advanced Statistical Physics"

Problem set 2, due June 5th, 2019

Problem 4) Mean field correlations in the Ising model

Consider an Ising modell on the d dimensional cubic lattice (i.e., square lattice in 2D, simple cubic in 3D etc.) with periodic boundary conditions and inhomogeneously fields

$$\mathcal{H} = -J \sum_{<ij>} S_i\, S_j - \sum_i h_i\, S_i. \tag{1}$$

In class, we have shown the exact relation $G_{ij} := \langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle = k_B T \frac{\partial \langle S_i \rangle}{\partial h_j}$. Using this result in the limit $h_i \to 0 \; \forall \, i$ allows us to calculate the correlations G_{ij}.

Here, we will use the Bragg Williams approximation. In this approximation, the mean magnetization at site i obeys the implicit equation.

$$m_i = \tanh[\beta(J \sum_{\text{Neighbors } j} m_j + h_i)]. \tag{2}$$

Consider the regime $T > T_c$ and the limit $h_i \to 0$. In this limit, Eq. (2) can be Taylor expanded for small m_i.

(a) Show that, in linear approximation, G_{ij} has the form

$$G_{ij} = G(\vec{r}_i - \vec{r}_j) = \frac{1}{(2\pi)^d} \int d\vec{p}\, \tilde{G}(\vec{p}) e^{i(\vec{r}_i - \vec{r}_j)\vec{p}} \quad \text{with} \quad \tilde{G}(\vec{p}) = \frac{1}{1 - \frac{\beta}{\beta_c} \frac{1}{q} \sum\limits_{\text{Neighbors } j} \cos(\vec{p}\vec{\tau}_j)}, \tag{3}$$

where the vector $\vec{r}_i$ is the position of site i and the vector $\vec{\tau}_j$ points from one site to the site of its jth neighbor. What is the integration volume $\int d\vec{p}$?

Hint: First establish a linear equation $h_i = \sum_j B_{ij} m_j$. This gives $\frac{\partial m_i}{\partial h_j} = B_{ij}^{-1}$. For symmetry reasons, the elements B_{ij} of the $(N \times N)$ matrix B depend only on $\vec{r}_i - \vec{r}_j$, i.e., $B_{ij} = B(\vec{r}_i - \vec{r}_j)$. Then diagonalize B by Fourier transform and invert it.

(b) Show: Close to T_c, the function $\tilde{G}(\vec{p})^{-1}$ can be expanded as

$$\tilde{G}(\vec{p})^{-1} \approx \frac{\beta}{\beta_c}(t + p^2 v(\hat{p})) \tag{4}$$

with $t = \beta_c/\beta - 1$, where v only depends on the unit vector $\hat{p}$ in the direction of $\vec{p}$. Calculate $v(\hat{p})$ for arbitrary dimensions.

(c) Consider specifically the one dimensional case. Calculate $G(x)$ using the approximation (4) and replacing the integration boundaries in $\int_{-\pi}^{\pi} dp...$ by $\int_{-\infty}^{\infty} dp...$ (use Residues). Alternatively, you may also try to calculate $G(x)$ directly from Eq. (3) without further approximations.

Show $G(x) \propto e^{-x/\xi}$ and calculate ξ as a function of t.

Problem 5) Ising antiferromagnet in Bragg-Williams approximation

Consider a two dimensional Ising antiferromagnet on the square lattice. The energy function is $\mathcal{H} = -J\sum_{<ij>} S_i\, S_j$ with $J < 0$. A suitable order parameter q for the antiferromagnetic order is the so-called staggered magnetization: We separate the lattice in checkerboard-manner into two sublattices a, b and define the sublattice magnetizations (per sublattice site) m_a and m_b. The total magnetization per site is then given by $m = (m_a + m_b)/2$ and the staggered magnetization by $q = (m_a - m_b)/2$.

Your task in this problem is to calculate the phase diagram in Bragg-Williams-approximation as a function of T and m.

(a) Determine the free energy per site, $F(q, m)/N$, for given q and m in Bragg Williams approximation.

Hint: To determine the Bragg Williams entropy of the two sublattices, the best way is to calculate it separately for each sublattice and then add it up.

(b) Consider the case $m = 0$. Minimize $F(m = 0, q)$ with respect to q and determine the temperature T_c of the phase transition. Compare your result with the result from Problem 3 and discuss your findings.

(c) Now consider the general case, $m \neq 0$. Argue that you must have $\frac{\partial^2 F}{\partial q^2}\Big|_{q=0} = 0$ at the phase transition. Use this to determine the location of the antiferromagnetic phase transition, T_c, as a function of m. You may use Mathematica.

Problem 6) Ising model with infinite range

Another example of an exactly solvable model with a phase transition is the Ising model with infinite range interactions, i.e., the energy function $\mathcal{H} = -\frac{J}{2N}\sum_{ij} S_i S_j - H\sum_i S_i$, where N is the number of spins and the sum runs over *all* spins.

(a) Show that the partition function can be written exactly as

$$Z_N = \sum_{M=-N}^{N} \binom{N}{\frac{N-M}{2}} \exp\left[\beta\left(\frac{J}{2N}M^2 + HM\right)\right] \tag{5}$$

with $M = \sum_i S_i = Nm$.

(b) For large N, you can approximate the binomial coefficient by the Stirling formula and the sum by an integral. Use this to rewrite the partition function in the form

$$Z_N = N\int_{-1}^{1} dm\, e^{-\beta N f(m)} \tag{6}$$

and give the explicit expression for $f(m)$.

(c) At $N \to \infty$, the integral (6) is dominated by the minimum m_0 of $f(m)$ and you can approximate $f(m) \approx f(m_0) + \frac{1}{2}f''(m_0)(m - m_0)^2$. Show that the free energy of the system is then given by $F = Nf(m_0)$ up to non-extensive corrections.

(d) Discuss your result. Show that the system has a phase transition and calculate the transition temperature. What critical exponents do you expect?