Exercises in "Advanced Statistical Physics"

Problem set 3, due June 19th, 2019

Choose two out of the three following problems.

Problem 7) Ginzburg Landau expansions for trigonal symmetry

Consider a system with a two dimensional order parameter $\vec{n} = (n_x, n_y)$ in a system with *trigonal* symmetry, i.e., invariance under rotation by an angle $\phi = 2\pi/3$. Construct the Landau expansion for this situation. Proceed as follows

(a) Construct the 2×2 rotation matrix D for this rotation. (obviously $D^3 = 1$ is the unit matrix.) For any given function $f(\vec{n})$, the function $g(\vec{n}) = f(\vec{n}) + f(D\vec{n}) + f(D^2\vec{n})$ is invariant under trigonal transformations. Use this to construct all invariants up to fourth order of n_x, n_y.

Example: To second order, we start by constructing all possible second order expressions in n_x, n_y, i.e., $f(\vec{n}) = n_x^2, n_y^2, n_x n_y$, and calculate the corresponding $g(\vec{n})$. The only non-vanishing outcome has the form $g(\vec{n}) \propto \vec{n}^2 = n_x^2 + n_y^2$.

(c) Use the invariants from (b) to construct a Landau expansion.

The result has the form

$$F/N = a + \frac{b}{2}\vec{n}^2 + \frac{c}{3} n_x(n_x^2 - 3n_y^2) + \frac{d}{3} n_y(n_y^2 - 3n_x)^2 + \frac{e}{4}(\vec{n}^2)^2 \qquad (1)$$

Do you expect a continuous phase transition or a first order phase transition?

(d) Now assume that, in addition, the system is invariant under sign reversal, $\vec{n} \leftrightarrow -\vec{n}$. The symmetry is then hexagonal, and some of the terms in the expansion above are forbidden. Which ones? What implications does this have for the nature of the phase transition?

(e) (Bonus) Construct the Landau expansion of the hexagonal system up to sixth order. Show that all invariants are fully isotropic up to fourth order $((\vec{n})^2$ and $(\vec{n})^4)$ and that you have only two linearly independent invariants at sixth order, the isotropic one $(\vec{n})^6$ and an anisotropic one, $A = n_x^6 - 15n_x^4 n_y^2 + 15 n_x^2 n_y^4 - n_y^6$.

Let Bennett Karetta tell you about my motivation to give you this problem.

Problem 8) **Modulated phases and Lifshitz point**

Consider a system with a one component order parameter $m(\vec{r})$. In class, we have introduced the Ginzburg-Landau functional $\mathcal{F}[m(\vec{r}] = \int \mathrm{d}\vec{r}\Big(f(m) + \frac{1}{2}g(\nabla m)^2\Big)$ with positive stiffness term, $g > 0$. We will now consider the case where g may change sign. Such Ginzburg-Landau expansions are used, e.g., to describe systems that exhibit patterned phases, stripe patterns. Examples are certain magnetic systems, certain polymer systems, and lipid-water mixtures.

Consider specifically

$$\mathcal{F}[m(\vec{r}] = \int \mathrm{d}\vec{r}\Big(\frac{b}{2}m^2 + \frac{c}{4}m^4 + \frac{g}{2}(\nabla m)^2 + \frac{v}{2}(\Delta m)^2 - hm\Big) \tag{2}$$

with $c, u > 0$.

(a) Minimize $\mathcal{F}[m(\vec{r})]$. The result is the differential equation

$$bm + cm^3 - g\Delta m + u\Delta^2 m. \tag{3}$$

(b) Neglect the term cm^3 for now. Carry out a Fourier transformation $m(\vec{r}) \to \tilde{m}(\vec{k})$, $h(\vec{r}) \to \tilde{h}(\vec{k})$. and show that Eq. (3) then turns into

$$b\tilde{m} + gk^2\tilde{m} + u(\vec{k}^2)^2\tilde{m} = \tilde{h}. \tag{4}$$

(c) Use Eq. (4) to determine the susceptibility $\tilde{\chi}(\vec{k}) = \delta\tilde{m}(\vec{k})/\delta\tilde{h}(\vec{k}$. Show that its maximum is always at $k = 0$ for $g > 0$, but it can have a maximum at $k \neq 0$ for $g < 0$. Determine the position k^* of that maximum.

(d) The susceptibility $\tilde{\chi}$ diverges at one value of b. Determine the critical point $b_c(g)$, where this happens for $g > 0$ and $g < 0$. For $g > 0$, the singularity marks an Ising-type transition. For $g < 0$, it marks a transition to a modulated structure with characteristic wave vector k^*. The point $g = 0$ is a special point called Lifshitz point. What happens with k^* at the Lifshitz point?

(e) Show that the susceptibility at the Lifshitz point can be written as $\tilde{\chi}(\vec{k}) \propto \frac{1}{b(1+k^4\xi^4)}$ with $\xi = (u/b)^{1/4}$. Assuming, as usual, $b \propto (T - T_c)$, the mean-field exponent ν at the Lifshitz point is thus given by $\nu = \nu_L = 1/4$, and the exponent γ by $\gamma = \gamma_L = 1$.

(f) The critical exponent β at the Lifshitz point is the same as that of the Ising model, $\beta_L = 1/2$ (Why?). Use this and the results from (e) to determine the upper critical dimension of the Lifshitz point.

Problem 9) Ginzburg-Landau theory of superconductors

In the Ginzburg Landau theory of superconductors, the order parameter is described by a complex function $\psi(\vec{r})$, where $|\psi(\vec{r})|^2/2$ is the density of the superconducting fraction of electrons. The free energy at given external magnetic field $\vec{H}$ is expanded in powers of ψ and a local, internal magnetic field $\vec{h} = \nabla \times \vec{A}$:

$$\mathcal{F}(\psi, \vec{A}) = \int d\vec{r} g \qquad \text{mit} \quad g = g_{n0} + \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{1}{2m^*}|(\frac{\hbar}{i}\nabla - \frac{e^*}{c}\vec{A})\psi|^2 + \frac{\vec{h}^2}{8\pi} - \frac{\vec{h}\vec{H}}{4\pi} \tag{5}$$

Here $e^* = 2e$ is the charge of an electron pair and $m^* \approx 2m_e$ their effective mass, the coefficients $\beta > 0$, α depend on the temperature, and the last two terms describe the free energy density of the magnetic field at fixed $\vec{H}$.

(a) Consider the homogeneous system ($\vec{h} = \text{const.}, \psi = \text{const.}$).
Sketch g for different values of α in the field free case $\vec{H} = 0$, $\vec{A} = 0$. Where is the phase transition? Minimize g and determine the value ψ_0 of the order parameter at the minimum as a function of α.

(b) Next consider $\vec{H} \neq 0$ and show that you must have $\vec{h} = 0$ also in this case in the superconductive case. Thus the superconductor expells the magnetic field. Calculate the free energy of the normal conductor (i.e., at $\psi = 0$). At which critical field H_c do you expect superconductivity to break down?

(c) Minimize $\mathcal{F}(\psi, \vec{A})$ with respect to ψ and $\vec{A}$ and derive the Ginzburg Landau equations. The solution is

$$\alpha\psi + \beta|\psi|^2\psi + \frac{1}{2m^*}(\frac{\hbar}{i}\nabla - \frac{e^*}{c}\vec{A})^2\psi = 0 \tag{6}$$

$$\frac{e^*\hbar}{2m^*i}(\psi^*\nabla\psi - \psi\nabla\psi^*) - \frac{e^{*2}}{m^*c}\vec{A}|\psi|^2 = \frac{c}{4\pi}(\nabla \times \vec{h}) \tag{7}$$

Together with the Maxwell equation $\vec{J}_s = \frac{c}{4\pi}\nabla\times\vec{h}$, Eq. (7) gives the superconducting current.

(d) Now require that the phase of ψ is constant, e.g., real-valued. With this assumption, you can easily find both important length scales of the system.
Start from equation (6) for the case $\vec{A} = 0$. Linearize the equation by expanding about ψ_0 (from (a)) up to first order. The result can be cast in the form $\xi^2\Delta f = 2f$ with $f = (\psi - \psi_0)/\psi_0$. This means that f is either zero throughout or decays exponentially (near surfaces), $f \propto e^{-r\sqrt{2}/\xi}$. Calculate ξ. What is the behavior of ξ close to the phase transition?
The length scale ξ is called *coherence length*.

Consider now the equation for the field, (7), in the case $\psi = \text{const}$. You can use it to derive an equation of the form $\lambda^2 \Delta \vec{h} = \vec{h}$. This means that $\vec{h}$ is either zero throughout or decays exponentially (near surfaces), $h \propto e^{-r/\lambda}$. Calculate λ. What is the behavior of λ close to the phase transition?

The length scale λ is called the *London penetration length*.

At domain boundaries between normal and superconducting phases, these two length scales compete with each other. An important dimensionless parameter is their ratio, $\kappa = \lambda/\xi$. Calculate κ. How does κ behave upon approaching the phase transition?

We will discuss the implications in more detail on the next exercise sheet.