Exercises in "Advanced Statistical Physics"

Problem set 4, due July 3rd, 2019

Problem 10) Scaling hypothesis for stretched polymers

Consider a polymer of length N (i.e., it is made of N monomers), which is stretched by applying a force f to both ends. Your task in the present problem is to derive the form of the force-extension curve $R_e \leftrightarrow f$ between the force f and the end-to-end extension in the direction of the force r_f, using scaling arguments.

The only length scale in this problem is the gyration radius R_g of the chains, which depends on N via $R_g = aN^\nu$, where $\nu \approx 3/5$ and a is some microscopic length. The intrinsic energy scale is the thermal energy, $k_B T$.

(a) Use dimension analysis to derive a general scaling hypothesis for the relation r_f as a function of f.

(b) Consider the limiting cases of strong and weak stretching. At strong stretching, you can assume $r_f \propto N$. At weak stretching, Hooke's law should hold, $r_f \propto f$.

How does r_f scale with f in the strong stretching case, how does it scale with N in the weak stretching case?

Problem 11) Ginzburg-Landau theory with Gaussian fluctuations

Consider a scalar Ginzburg-Landau theory with scalar order parameter $\Phi(\vec{r})$ and the Ginzburg-Landau functional

$$\mathcal{F}[\Phi(\mathbf{r})] = \int \mathrm{d}^d r \, \Big(\frac{1}{2}(\nabla\Phi)^2 + f(\Phi)\Big) \tag{1}$$

in a finite box of length L (volume $V = L^d$) with periodic boundary conditions.

Evaluate the partition function of this system in saddle point approximation, including Gaussian fluctuations.

(a) Show that the saddle field (the field $\bar{\Phi}(\vec{r})$ that minimizes $\mathcal{F}$) satisfies the equation $-\Delta\bar{\Phi} + f'(\bar{\Phi}) \equiv 0$. Thus the homogeneous solution is given by $\Phi(\vec{r}) \equiv \bar{\Phi}_0$ with $\bar{\Phi}_0$ defined such that $f'(\bar{\Phi}_0) = 0$.

Next consider small deviations $\Phi(\vec{r}) = \bar{\Phi}_0 + \eta(\vec{r})$ (with small $\eta(\vec{r})$ and expand $\mathcal{F}[\Phi(\vec{r})]$ up to second order in $\eta(\vec{r})$.

(b) To proceed, it is convenient to switch to the Fourier represenation of $\eta(\vec{r})$, which we define as $\tilde{\eta}_{\vec{k}} = \int \mathrm{d}^d r \, \mathrm{e}^{i\vec{k}\cdot\vec{r}} \eta(\vec{r})$, where the components of the $\vec{k}$ are multiples of $2\pi/L$. (The inverse tranform is $\eta(\vec{r}) = \frac{1}{V}\sum_{\vec{k}} \mathrm{e}^{-i\vec{k}\cdot\vec{r}} \tilde{\eta}_{\vec{k}}$.)

Calculate the Ginzburg-Landau free energy as a functional of $\tilde{\eta}_{\vec{k}}$, i.e., $\mathcal{F}[\tilde{\eta}_{\vec{k}}]$. The result is

$$\mathcal{F}[\tilde{\eta}_{\vec{k}}] = V f(\bar{\Phi}_0) + \frac{1}{2V}\sum_{\vec{k}} |\tilde{\eta}_{\vec{k}}|^2 (k^2 + \xi^{-2}) \tag{2}$$

with $\xi^{-2} := f''(\bar{\Phi}_0)$.

(c) Based on the approximation (2), you can now evaluate the partition function and calculate the free energy. Use the relation $\int \mathcal{D}\Phi(\vec{r}) \cdots = \prod'_{\vec{k}} \int \frac{1}{V v_0} \mathrm{d}^2 \tilde{\eta}_{\vec{k}} \cdots$, where $\int \mathrm{d}^2 \tilde{\eta}_{\vec{k}} \cdots$ denotes the integral over the whole complex plane, v_0 is a microscopic volume, and $\prod'_{\vec{k}}$ runs only over half of the possible $\vec{k}$-vectors, since $\tilde{\eta}_{\vec{k}}$ and $\tilde{\eta}_{-\vec{k}} = \tilde{\eta}^*_{\vec{k}}$ are not independent degrees of freedom.

The result has the form $F = V(f(\bar{\Phi}_0) + g(\bar{\Phi}_0))$ with $g(\bar{\Phi}_0) = \frac{1}{2\beta V} \sum_{\vec{k}} \ln C(k^2 + f''(\bar{\Phi}_0))$ (C is a constant). $g(\bar{\phi}_0)$ is the fluctuation correction to the free energy. Since the Gaussian fluctuations are distributed symmetrically about $\bar{\Phi}_0$, they do not affect the statistical averages $\langle \Phi \rangle = \bar{\Phi}_0$, but they do contribute to the free energy and its derivatives, i.e., the specific heat.

Problem 12) <u>Renormalization of the one-dimensional Ising model</u>

The renormalization group (RG) program can be carried out exactly for the one dimensional Ising model. Consider a chain of $N = 2^n$ Ising spins with periodic boundary conditions and the Hamiltonian

$$\beta \mathcal{H}_{g,h,K}[S_i] = gN - h\frac{1}{2}\sum_{i=1}^{N}(S_i + S_{i+1}) - K\sum_{i=1}^{N} S_i S_{i+1}, \tag{3}$$

where gN is a constant term.

(a) In the first step of a RG transformation, the degrees of freedom must be *thinned out*. Here, we do this by averaging over every second spin, i.e., replacing every block of two consecutive spins by the value of the second spin. Calculate the resulting partition function. Your result is

$$\mathcal{Z} = \sum_{S_2, S_4, S_6, \ldots} \prod_{i=2,4,6,\ldots} \left[2\mathrm{e}^{-2\beta g} \mathrm{e}^{\frac{1}{2}\beta h(S_i + S_{i+2})} \cosh(\beta(h + K(S_i + S_{i+2}))\right] \tag{4}$$

(b) The next step is to *rescale* the theory. To this end, define $S'_i = S_{2i}$ and rewrite $\mathcal{Z}$ in the form $\mathcal{Z} = \sum_{S'_i} \mathrm{e}^{-\beta \mathcal{H}_{g',h',K'}[S'_i]}$, where $\mathcal{H}$ has the functional form of Eq. (3).

<u>Hint:</u> For every factor $[\cdots]$ in Eq. (4), the sum $(S_{2j} + S_{2j+2})$ can only take three different values. Use this to rewrite the factors in the square brackets in Eq. (4) as $\left[\cdots\right] = \left[\mathrm{e}^{\beta(2g' + \frac{1}{2}h'(S'_j + S'_{j+1}) + K' S'_j S'_{j+1})}\right]$.

The term $g'(K,h)$ gives a regular contribution to the free energy. The transformation $(K', h') = R(K, h)$ defines the RG transformation.

(c) Find the fixed points of the RG transformation. You should find a set of trivial fixed points and one critical fixed point at $K^* \to \infty, h^* = 0$. What is the interpretation of this result?

(d) Substitute $x = \mathrm{e}^{-4K}$. The critical point is then at $x^* = 0, h^* = 0$. Linearize the RG equations about the fixed point and determine the critical exponents y_x and y_h associated with the fields x and h. What is the resulting scaling form for the singular part of the free energy of the 1D Ising model?

The result is $f_s(x, h) = x^{1/2} F_f(h/x^{1/2})$.