Dynamic Processes of Financial Markets and the Black-Scholes Model

Higher Statistical Mechanics

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27.5.2019

Introduction

- in both physics and finance observe dynamical systems
- experiments and markets produce lots of data

source:
figure 1:https://commons.wikimedia.org/wiki/File:Wall_Street_(5899300483).jpg
figure 2:https://commons.wikimedia.org/wiki/File:CERN_(7825770258).jpg

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- theoretical physics and financial mathematics

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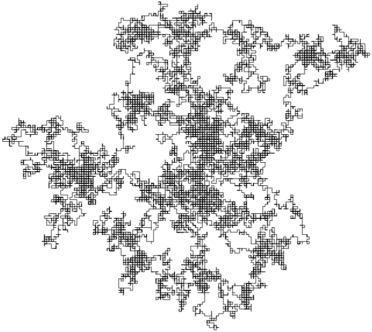

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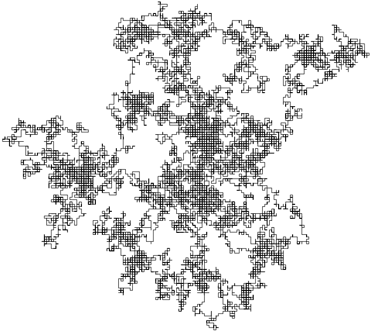

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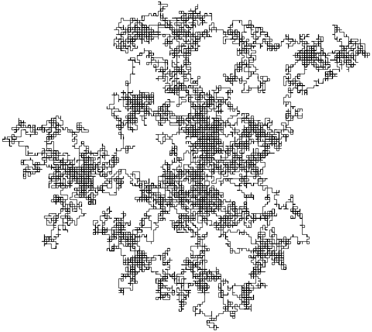
  - $\rightsquigarrow$ need basics in stochastic and finance



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- how does pricing for financial instruments work?

Basics in Financial Markets

- on an exchange financial instruments are traded

pricing of cash instruments

- buyers and sellers write orders in **order book**
- price determined by exchange **specific rules** and entries of order book

source:https://commons.wikimedia.org/wiki/File:Sao_Paulo_Stock_Exchange.jpg

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- investors try to make profit with **long term ownership**

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definition: Wiener process (Brownian motion)

a real stochastic process $X = (X_t, t \in I)$ is called Wiener process iff:

$$\begin{cases} X_0 = 0 \text{ almost everywhere} \\ \text{for } 0 < t \leq t' : X_{t+t'} - X_t \text{ is independent of } X_s \text{ for } s < t \\ X_{t+t'} - X_t \propto \mathcal{N}(0, t') \\ X_t \text{ is continuous in } t \text{ almost everywhere} \end{cases}$$

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definition: geometric Brownian motion

let X be a Wiener process then $W_t = a \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma X_t\right)$ with $a, \sigma, \mu \in \mathbb{R}$ is called geometric Brownian motion, μ is called drift and σ volatility

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7. assets are traded in arbitrary fractions
8. the market is arbitrage free

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under assumptions made, deposit value with a starting value $\Pi(0)$ and interest rate r is :

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where we used:

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eliminate dS and plug in Π:

$$\frac{\partial O}{\partial t} + \frac{1}{2}(\sigma S(t))^2\frac{\partial^2 O}{\partial S^2} + rS(t)\frac{\partial O}{\partial S} - rO = 0$$

the Black-Scholes equation

Discussion of the Black-Scholes Model

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Limitations of the Black-Scholes Model

stock price performs a geometric Brownian motion, but:

source: https://www.ariva.de/

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$\rightsquigarrow$ underestimation of extreme moves, violates assumption 4

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$\rightsquigarrow$ Black-Scholes model can serve as basis for more refined model

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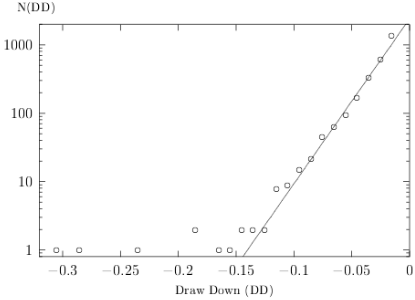

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draw down: relative loss from local maximum to next local minimum

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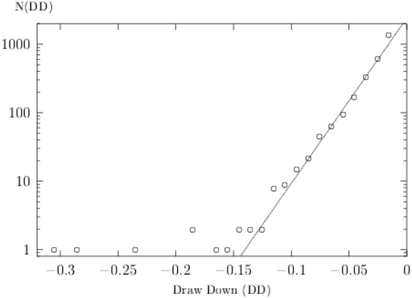

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change of exponential behaviour can be explained by a change of market dynamics

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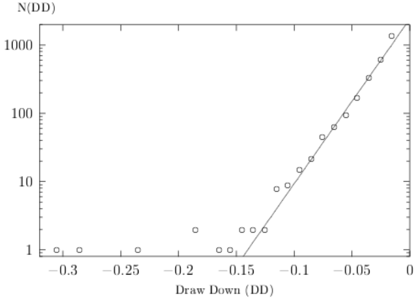

source:https://arxiv.org/abs/cond-mat/9712005v3

draw down: relative loss from local maximum to next local minimum

change of exponential behaviour can be explained by a change of market dynamics

$\rightsquigarrow$ phase transition

Financial Crashes

source:https://www.ariva.de/

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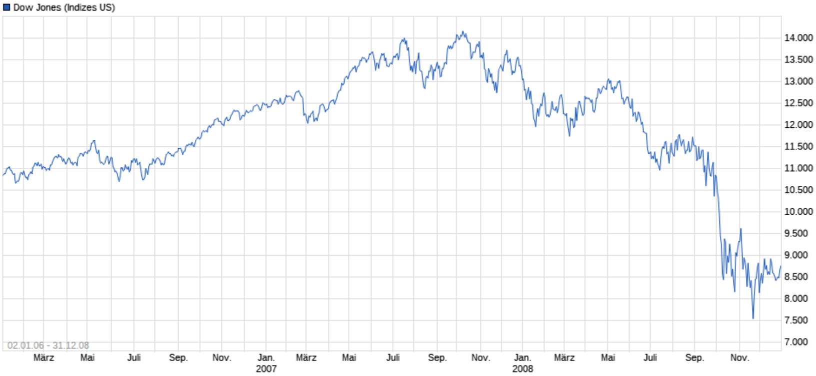
$\rightsquigarrow$ earthquake models

Financial Crashes

not all crashes are earthquake like



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- decision based on rational assessment and environment (information from colleagues and market)

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if big cluster emerges and decides to sell $\rightsquigarrow$ crash

Summary

- financial markets and assets can be modelled with statistic methods
- limited data
- unknown micro dynamics
- models have restricted predictability

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