Noli turbare circulos meos! - Packalgorithmen für Kreise

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Archimedes von Syrakus (287 v. Chr. - 212 v. Chr.)

Überblick

1. Das Apollonische Problem
2. Die Lösung von Viète
3. Kleinster einschließender Kreis für Kreise
4. Ein Kreispackungsproblem
5. Ein einfacher Packalgorithmus
6. Eine industrielle Anwendung

Apollonius von Perge (262 v. Chr. - 190 v. Chr.)

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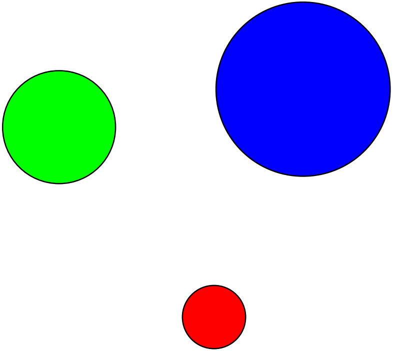

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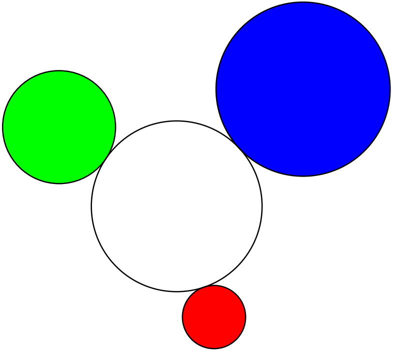

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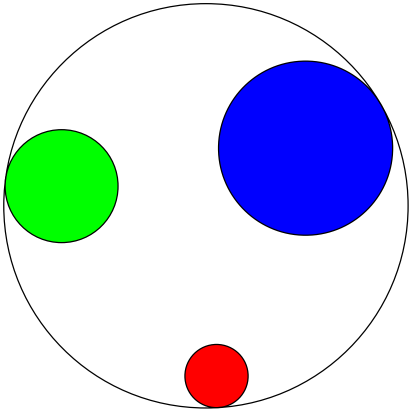

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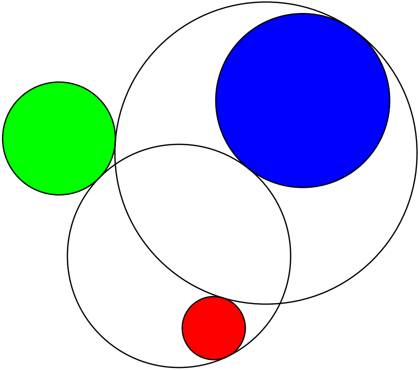

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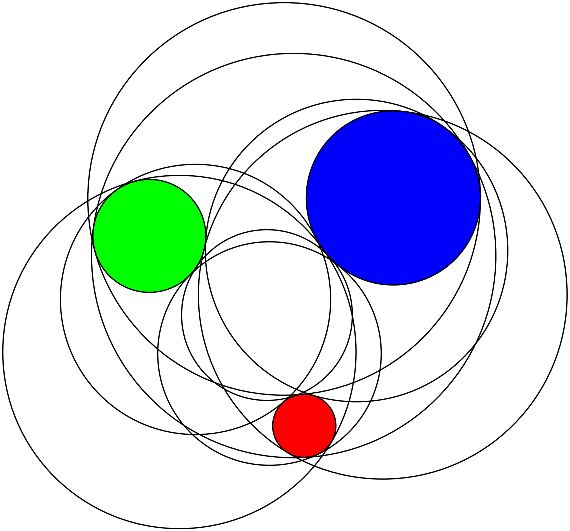
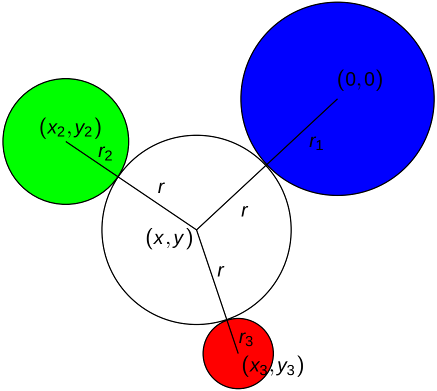


Die Lösung von Viète



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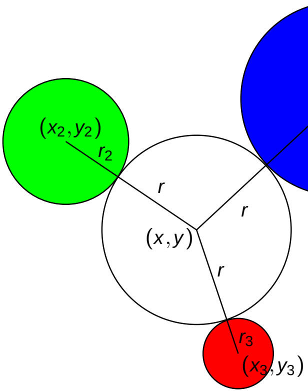
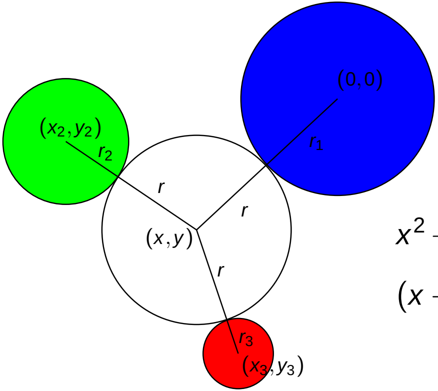


$$x^2 + y^2 = (r + r_1)^2$$

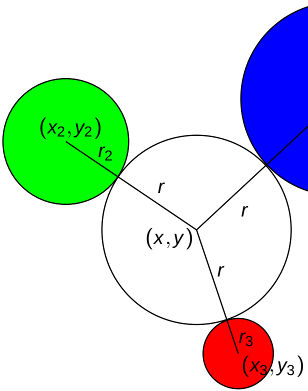
Die Lösung von Viète



$$x^2+y^2=(r+r_1)^2$$

$$(x-x_2)^2+(y-y_2)^2=(r+r_2)^2$$

Die Lösung von Viète



$$x^2 + y^2 = (r + r_1)^2$$

$$(x - x_2)^2 + (y - y_2)^2 = (r + r_2)^2$$

$$(x - x_3)^2 + (y - y_3)^2 = (r + r_3)^2$$

Die Lösung von Viète

$$
\begin{aligned}
x^2 + y^2 &= (r + r_1)^2 \\
(x - x_2)^2 + (y - y_2)^2 &= (r + r_2)^2 \\
(x - x_3)^2 + (y - y_3)^2 &= (r + r_3)^2
\end{aligned}
$$

Die Lösung von Viète

$$\begin{aligned} x^2 + y^2 &= (r + r_1)^2 \\ (x - x_2)^2 + (y - y_2)^2 &= (r + r_2)^2 \\ (x - x_3)^2 + (y - y_3)^2 &= (r + r_3)^2 \end{aligned}$$

$$x^2 + y^2 = r^2 + 2r\, r_1 + r_1^2$$

Die Lösung von Viète

$$\begin{aligned}
x^2 + y^2 &= (r + r_1)^2 \\
(x - x_2)^2 + (y - y_2)^2 &= (r + r_2)^2 \\
(x - x_3)^2 + (y - y_3)^2 &= (r + r_3)^2
\end{aligned}$$

$$\begin{aligned}
x^2 + y^2 &= r^2 + 2r\,r_1 + r_1^2 \\
x^2 - 2x_2 x + x_2^2 + y^2 - 2y_2 y + y_2^2 &= r^2 + 2r\,r_2 + r_2^2
\end{aligned}$$

Die Lösung von Viète

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(x - x_3)^2 + (y - y_3)^2 &= (r + r_3)^2
\end{aligned}
$$

$$
\begin{aligned}
x^2 + y^2 &= r^2 + 2r\,r_1 + r_1^2 \\
x^2 - 2x_2\,x + x_2^2 + y^2 - 2y_2\,y + y_2^2 &= r^2 + 2r\,r_2 + r_2^2 \\
x^2 - 2x_3\,x + x_3^2 + y^2 - 2y_3\,y + y_3^2 &= r^2 + 2r\,r_3 + r_3^2
\end{aligned}
$$

Die Lösung von Viète

$$\begin{aligned} x^2 + y^2 &= (r + r_1)^2 \\ (x - x_2)^2 + (y - y_2)^2 &= (r + r_2)^2 \\ (x - x_3)^2 + (y - y_3)^2 &= (r + r_3)^2 \end{aligned}$$

$$\begin{aligned} x^2 + y^2 &= r^2 + 2r\,r_1 + r_1^2 \\ x^2 - 2x_2 x + x_2^2 + y^2 - 2y_2 y + y_2^2 &= r^2 + 2r\,r_2 + r_2^2 \\ x^2 - 2x_3 x + x_3^2 + y^2 - 2y_3 y + y_3^2 &= r^2 + 2r\,r_3 + r_3^2 \end{aligned}$$

Die Lösung von Viète

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\begin{aligned}
x^2 + y^2 &= (r + r_1)^2 \\
(x - x_2)^2 + (y - y_2)^2 &= (r + r_2)^2 \\
(x - x_3)^2 + (y - y_3)^2 &= (r + r_3)^2
\end{aligned}
$$

$$
\begin{aligned}
{\color{red}x^2} + {\color{red}y^2} &= {\color{red}r^2} + 2r\,r_1 + r_1^2 \\
{\color{red}x^2} - 2x_2\,x + x_2^2 + {\color{red}y^2} - 2y_2\,y + y_2^2 &= {\color{red}r^2} + 2r\,r_2 + r_2^2 \\
{\color{red}x^2} - 2x_3\,x + x_3^2 + {\color{red}y^2} - 2y_3\,y + y_3^2 &= {\color{red}r^2} + 2r\,r_3 + r_3^2
\end{aligned}
$$

$$
2x_2\,x - x_2^2 + 2y_2\,y - y_2^2 = 2r\,(r_1 - r_2) + r_1^2 - r_2^2
$$

Die Lösung von Viète

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\begin{aligned}
x^2 + y^2 &= (r + r_1)^2 \\
(x - x_2)^2 + (y - y_2)^2 &= (r + r_2)^2 \\
(x - x_3)^2 + (y - y_3)^2 &= (r + r_3)^2
\end{aligned}
$$

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\begin{aligned}
{\color{red}x^2} + {\color{red}y^2} &= {\color{red}r^2} + 2r\,r_1 + r_1^2 \\
{\color{red}x^2} - 2x_2\,x + x_2^2 + {\color{red}y^2} - 2y_2\,y + y_2^2 &= {\color{red}r^2} + 2r\,r_2 + r_2^2 \\
{\color{red}x^2} - 2x_3\,x + x_3^2 + {\color{red}y^2} - 2y_3\,y + y_3^2 &= {\color{red}r^2} + 2r\,r_3 + r_3^2
\end{aligned}
$$

$$
\begin{aligned}
2x_2\,x - x_2^2 + 2y_2\,y - y_2^2 &= 2r\,(r_1 - r_2) + r_1^2 - r_2^2 \\
2x_3\,x - x_3^2 + 2y_3\,y - y_3^2 &= 2r\,(r_1 - r_3) + r_1^2 - r_3^2
\end{aligned}
$$

$$\begin{aligned}
x^2+y^2 &= (r+r_1)^2 \\
(x-x_2)^2+(y-y_2)^2 &= (r+r_2)^2 \\
(x-x_3)^2+(y-y_3)^2 &= (r+r_3)^2
\end{aligned}$$

$$\begin{aligned}
x^2+y^2 &= r^2+2r\,r_1+r_1^2 \\
x^2-2x_2x+x_2^2+y^2-2y_2y+y_2^2 &= r^2+2r\,r_2+r_2^2 \\
x^2-2x_3x+x_3^2+y^2-2y_3y+y_3^2 &= r^2+2r\,r_3+r_3^2
\end{aligned}$$

$$\begin{aligned}
2x_2x-x_2^2+2y_2y-y_2^2 &= 2r\,(r_1-r_2)+r_1^2-r_2^2 \\
2x_3x-x_3^2+2y_3y-y_3^2 &= 2r\,(r_1-r_3)+r_1^2-r_3^2
\end{aligned}$$

$$x_2x+y_2y = r\,(r_1-r_2)+\frac{1}{2}(r_1^2-r_2^2+x_2^2+y_2^2)$$

Die Lösung von Viète

$$\begin{aligned}
x^2 + y^2 &= (r + r_1)^2 \\
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\end{aligned}$$

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x^2 + y^2 &= r^2 + 2r\,r_1 + r_1^2 \\
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x^2 - 2x_3 x + x_3^2 + y^2 - 2y_3 y + y_3^2 &= r^2 + 2r\,r_3 + r_3^2
\end{aligned}$$

$$\begin{aligned}
2x_2 x - x_2^2 + 2y_2 y - y_2^2 &= 2r\,(r_1 - r_2) + r_1^2 - r_2^2 \\
2x_3 x - x_3^2 + 2y_3 y - y_3^2 &= 2r\,(r_1 - r_3) + r_1^2 - r_3^2
\end{aligned}$$

$$\begin{aligned}
x_2 x + y_2 y &= r\,(r_1 - r_2) + \frac{1}{2}(r_1^2 - r_2^2 + x_2^2 + y_2^2) \\
x_3 x + y_3 y &= r\,(r_1 - r_3) + \frac{1}{2}(r_1^2 - r_3^2 + x_3^2 + y_3^2)
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Die Lösung von Viète

$$\begin{aligned} x_2 x + y_2 y &= r(r_1 - r_2) + \frac{1}{2}(r_1^2 - r_2^2 + x_2^2 + y_2^2) \\ x_3 x + y_3 y &= r(r_1 - r_3) + \frac{1}{2}(r_1^2 - r_3^2 + x_3^2 + y_3^2) \end{aligned}$$

$$\begin{aligned} x_2 x + y_2 y &= d_2 \\ x_3 x + y_3 y &= d_3 \end{aligned}$$

Die Lösung von Viète

$$\begin{aligned} x_2 x + y_2 y &= r(r_1 - r_2) + \frac{1}{2}(r_1^2 - r_2^2 + x_2^2 + y_2^2) \\ x_3 x + y_3 y &= r(r_1 - r_3) + \frac{1}{2}(r_1^2 - r_3^2 + x_3^2 + y_3^2) \end{aligned}$$

$$\begin{aligned} x_2 x + y_2 y &= d_2 \\ x_3 x + y_3 y &= d_3 \end{aligned}$$

$$x = \frac{d_2 y_3 - d_3 y_2}{x_2 y_3 - x_3 y_2} \qquad y = \frac{x_2 d_3 - x_3 d_2}{x_2 y_3 - x_3 y_2}$$

Die Lösung von Viète

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x_3 x + y_3 y &= r(r_1 - r_3) + \frac{1}{2}(r_1^2 - r_3^2 + x_3^2 + y_3^2)
\end{aligned}
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\begin{aligned}
x_2 x + y_2 y &= d_2 \\
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x = \frac{d_2 y_3 - d_3 y_2}{x_2 y_3 - x_3 y_2} \qquad y = \frac{x_2 d_3 - x_3 d_2}{x_2 y_3 - x_3 y_2}
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$$
x^2 + y^2 = r^2 + 2r\, r_1 + r_1^2
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Die Lösung von Viète

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\begin{aligned}
x_2 x + y_2 y &= r\,(r_1 - r_2) + \frac{1}{2}(r_1^2 - r_2^2 + x_2^2 + y_2^2) \\
x_3 x + y_3 y &= r\,(r_1 - r_3) + \frac{1}{2}(r_1^2 - r_3^2 + x_3^2 + y_3^2)
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$$

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\begin{aligned}
x_2 x + y_2 y &= d_2 \\
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x = \frac{d_2 y_3 - d_3 y_2}{x_2 y_3 - x_3 y_2} \qquad y = \frac{x_2 d_3 - x_3 d_2}{x_2 y_3 - x_3 y_2}
$$

$$
x^2 + y^2 = r^2 + 2r\,r_1 + r_1^2
$$

$$
\Longrightarrow \qquad \alpha r^2 + \beta r + \gamma = 0
$$

Kleinster einschließender Kreis für Kreise

$R = \infty$

Für alle Tripel (i, j, k) mit $i < j < k$:

$A =$ Apollonius Kreis für die Kreise i, j, k

Wenn alle Kreise in A enthalten

Wenn $radius(A) < R$

$R = radius(A)$

Kleinster einschließender Kreis für Kreise

$R = \infty$

Für alle Tripel (i,j,k) mit $i < j < k$:

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Wenn alle Kreise in A enthalten

Wenn $radius(A) < R$

$R = radius(A)$

Für alle Paare (i,j) mit $i < j$:

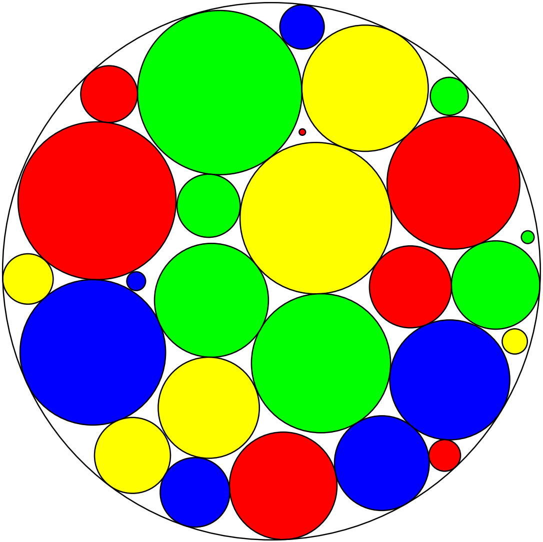
$A =$ Apollonius Kreis für die Kreise i,j

Wenn alle Kreise in A enthalten

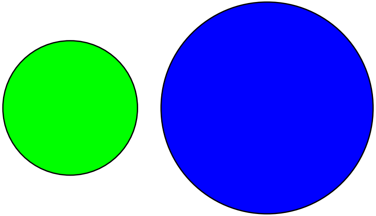
Wenn $radius(A) < R$

$R = radius(A)$

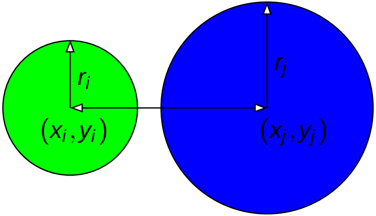
Kreispackungen



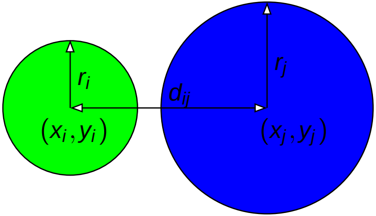
Kontaktsimulation



Kontaktsimulation

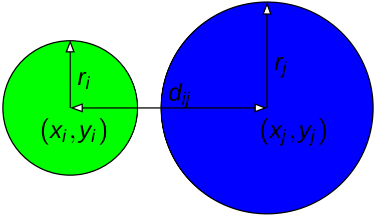


Kontaktsimulation



$$d_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$$

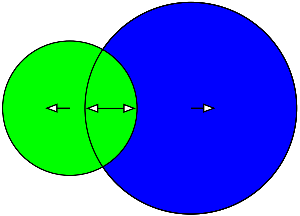
Kontaktsimulation



$$d_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$$

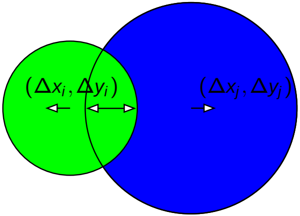
$d_{ij} > r_i + r_j \quad \Longrightarrow \quad$ keine Kollision

Kontaktsimulation



Kontaktsimulation

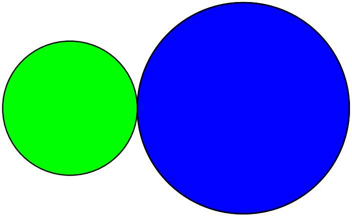
$d_{ij} < r_i + r_j \quad \Longrightarrow \quad$ Kollision



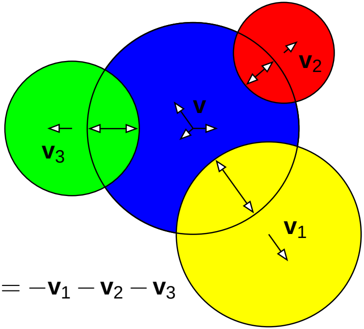
$$\Delta x_i = \frac{r_i + r_j - d_{ij}}{2} \, \frac{x_i - x_j}{d_{ij}} = -\Delta x_j$$

$$\Delta y_i = \frac{r_i + r_j - d_{ij}}{2} \, \frac{y_i - y_j}{d_{ij}} = -\Delta y_j$$

Kontaktsimulation

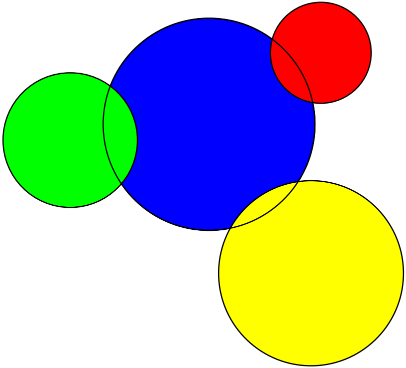


Kontaktsimulation



$$\mathbf{v} = -\mathbf{v}_1 - \mathbf{v}_2 - \mathbf{v}_3$$

Kontaktsimulation



Ein einfacher Algorithmus zur Kontaktsimulation

Für alle Kreise i:

$$x_i' = x_i\,, \quad y_i' = y_i$$

Für alle Paare (i,j) mit $i < j$:

Wenn $d_{ij} < r_i + r_j$

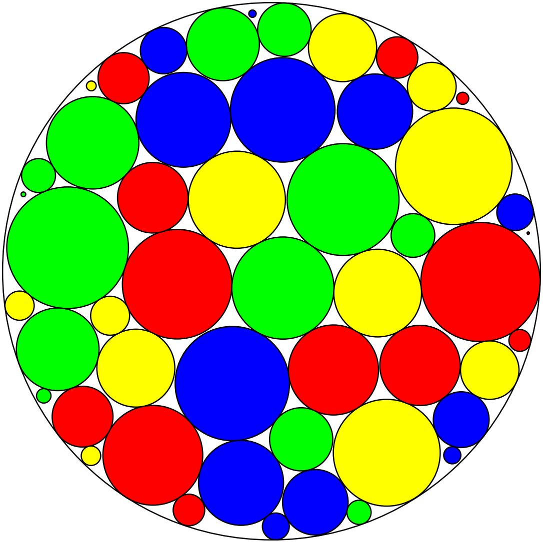
$$x_i' = x_i' + \Delta x_i\,, \quad y_i' = y_i' + \Delta y_i$$

$$x_i' = x_j' + \Delta x_j\,, \quad x_j' = y_j' + \Delta y_j$$

Für alle Kreise i:

$$x_i = x_i'\,, \quad y_i = y_i'$$

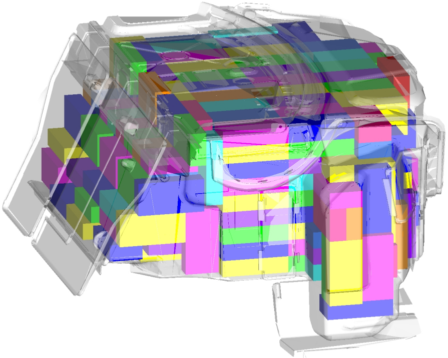
Beste Packung für 50 Kreise



Kofferraumausliterung

Thema: Automatische Bestimmung des Kofferraumvolumens nach DIN 70020 und SAE J1100

Methoden: Kombinatorische graphtheoretische Algorithmen, Monte-Carlo Simulation, Kontaktsimulation



Kooperationspartner: Abt. Gesamtfahrzeug Konstruktion, DaimlerChrysler AG in Sindelfingen