Ab initio molecular dynamics: Basic Theory and Advanced Methods

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Marialore Sulpizi Ab initio molecular dynamics: Basic Theory and Advanced Met

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Outline

Unifying molecular dynamics and electronic structure Car-Parrinello molecular dynamics The Local-Density Approximation

Unifying molecular dynamics and electronic structure Route 1 Route 2

Car-Parrinello molecular dynamics

Car-Parrinello Lagrangian and equations of motion Thermostat on the electrons Analytic and numerical error estimates CP vs BO

The Local-Density Approximation

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Route 1 Route 2

Unifying molecular dynamics and electronic structure

Starting point: the non-relativistic Schrödinger equation for electron and nuclei.

$$i\hbar\frac{\partial}{\partial t}\Phi(\{\mathbf{r}_i\},\{\mathbf{R}_i\};t) = H\Phi(\{\mathbf{r}_i\},\{\mathbf{R}_i\};t)$$
(1)

where H is the standard Hamiltonian:

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► Route 1:

1) The electronic problem is solved in the *time-independent* Schrodinger eq.

2) from here the adiabatic approximation for the nuclei is derived, and as a special case the Born-Oppenheimer dynamics.

3) The classical limit leads then to the classical molecular dynamics for the nuclei.

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Route 1 Route 2

Goal: derive the classical molecular dynamics. As intermediate two variant of *ab initio* molecular dynamics are derived.

Solve the electronic part for fixed nuclei

$$H_e(\lbrace \mathbf{r}_i \rbrace, \lbrace \mathbf{R}_i \rbrace) \Psi_k = E_k(\lbrace \mathbf{R}_i \rbrace) \Psi_k(\lbrace \mathbf{r}_i \rbrace, \lbrace \mathbf{R}_i \rbrace)$$
(6)

where $\Psi_k({\mathbf{r}_i}, {\mathbf{R}_i})$ are a set of orthonormal solutions, satisphying:

$$\int \Psi_k^*(\{\mathbf{r}_i\}, \{\mathbf{R}_i\}) \Psi_l(\{\mathbf{r}_i\}, \{\mathbf{R}_i\}) d\mathbf{r} = \delta_{kl}$$
(7)

Knowing the *adiabatic solutions* to the electronic problem, the total wavefunction can be expanded as:

$$\Phi(\{\mathbf{r}_i\}, \{\mathbf{R}_i\}, t) = \sum_{l=0}^{\infty} \Psi_l(\{\mathbf{r}_i\}, \{\mathbf{R}_i\}) \chi_l(\{\mathbf{R}_i\}, t)$$
(8)

Eq.8 is the *ansatz* introduced by Born in 1951.

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Inserting (8) into Eq.(1) we obtain:

$$\left[-\sum_{I}\frac{\hbar^{2}}{2M_{I}}\nabla_{I}^{2}+E_{k}(\{\mathbf{R}_{i}\})\right]\chi_{k}+\sum_{I}C_{kI}\chi_{I}=i\hbar\frac{\partial}{\partial t}\chi_{k}$$
(9)

where:

$$C_{kl} = \int \Psi_k^* \left[-\sum_I \frac{\hbar^2}{2M_I} \nabla_I^2 \right] \Psi_I d\mathbf{r} +$$

$$+ \frac{1}{M_I} \sum_I \{ \int \Psi_k^* [-i\hbar \nabla_I] \Psi_I d\mathbf{r} \} [-i\hbar \nabla_I]$$
(11)

is the exact non-adiabatic coupling operator.

The adiabatic approximation to the full problem Eq.9 is obtained considering only the diagonal terms:

$$C_{kk} = -\sum_{I} \frac{\hbar^2}{2M_I} \int \Psi_k^* \nabla_I^2 \Psi_k d\mathbf{r}$$
(12)

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$$\left[-\sum_{I}\frac{\hbar^{2}}{2M_{I}}\nabla_{I}^{2}+E_{k}(\{\mathbf{R}_{i}\})+C_{kk}(\{\mathbf{R}_{i}\})\right]\chi_{k}=i\hbar\frac{\partial}{\partial t}\chi_{k} \qquad (13)$$

The motion of the nuclei proceed without changing the quantum state, k, of the electronic subsystem during the evolution. The coupled wavefunction can be simplified as:

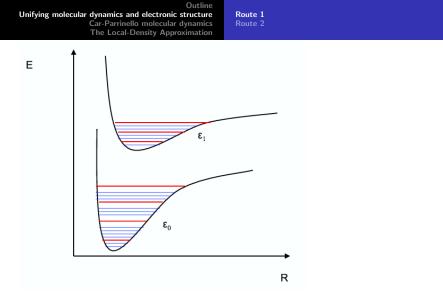
$$\Phi(\{\mathbf{r}_i\}, \{\mathbf{R}_i\}, t) \approx \Psi_k(\{\mathbf{r}_i\}, \{\mathbf{R}_i\})\chi_k(\{\mathbf{R}_i\}, t)$$
(14)

the ultimate simplification consist in neglecting also the correction term $C_{kk}({\mathbf{R}_i})$, so that

$$\left[-\sum_{I}\frac{\hbar^{2}}{2M_{I}}\nabla_{I}^{2}+E_{k}(\{\mathbf{R}_{i}\})\right]\chi_{k}=i\hbar\frac{\partial}{\partial t}\chi_{k}$$
(15)

This is the Born-Oppenheimer approximation.

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On each Born-Oppenheimer surface, the nuclear eigenvalue problem can be solved, which yields a set of levels (rotational and vibrational in the nuclear motion) as illustrated in the figure. $\langle \Box \rangle \langle \Box \rangle \langle \Box \rangle \langle \Xi \rangle \langle \Xi \rangle \rangle \geq 0$

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The next step is to derive the classical molecular dynamics for the nuclei. The route we take is the following:

$$\chi_k(\{\mathbf{R}_i\}, t) = A_k(\{\mathbf{R}_i\}, t) \exp[iS_k(\{\mathbf{R}_i\}, t)/\hbar]$$
(16)

the amplitude A_k and the phase S_k are both real and $A_k > 0$. Next we subistutute the expression for χ_k into eq. 15.

$$\left[-\sum_{I}\frac{\hbar^{2}}{2M_{I}}\nabla_{I}^{2}+E_{k}(\{\mathbf{R}_{i}\})\right]A_{k}(\{\mathbf{R}_{i}\},t)exp[iS_{k}(\{\mathbf{R}_{i}\},t)/\hbar]=$$
(17)

$$i\hbar \frac{\partial}{\partial t} \left(A_k(\{\mathbf{R}_i\}, t) exp[iS_k(\{\mathbf{R}_i\}, t)/\hbar] \right)$$
 (18)

$$-\sum_{I}\frac{\hbar^{2}}{2M_{I}}\nabla_{I}\left(\nabla_{I}\left(A_{k}exp[iS_{k}/\hbar]\right)\right)+E_{k}A_{k}exp[iS_{k}/\hbar]=$$
(19)

$$i\hbar \frac{\partial A_k}{\partial t} \exp[iS_k/\hbar] + i\hbar \frac{i}{\hbar} \frac{\partial S_k}{\partial t} A_k \exp[iS_k/\hbar]$$
(20)

$$-\sum_{I}\frac{\hbar^{2}}{2M_{I}}\nabla_{I}\left(\nabla_{I}A_{k}exp[iS_{k}/\hbar]+A_{k}\frac{i}{\hbar}\nabla S_{k}exp[iS_{k}/\hbar]\right)+E_{k}A_{k}exp[iS_{k}/\hbar]=$$
(21)

$$i\hbar \frac{\partial A_k}{\partial i} exp[iS_k/\hbar] - \frac{\partial S_k}{\partial i} A_k exp[iS_k/\hbar]$$
 (22)

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$$-\sum_{I} \frac{\hbar^{2}}{2M_{I}} \nabla_{I}^{2} A_{k} - \sum_{I} \frac{\hbar^{2}}{2M_{I}} \nabla_{I} A_{k} \frac{i}{\hbar} \nabla_{I} S_{k} - \sum_{I} \frac{\hbar^{2}}{2M_{I}} \nabla_{I} A_{k} \frac{i}{\hbar} \nabla_{I} S_{k}$$
(23)
$$-\sum_{I} \frac{\hbar^{2}}{2M_{I}} A_{k} \left(\frac{i}{\hbar}\right)^{2} (\nabla S_{k})^{2} - \sum_{I} \frac{\hbar^{2}}{2M_{I}} A_{k} \frac{i}{\hbar} \nabla^{2} S_{k} + E_{k} A_{k} =$$
(24)
$$i\hbar \frac{\partial A_{k}}{\partial t} - \frac{\partial S_{k}}{\partial t} A_{k}$$
(25)

Separating the real and the imaginary parts, we obtain:

$$\frac{\partial S_k}{\partial t} + \sum_I \frac{1}{2M_I} (\nabla S_k)^2 + E_k = -\sum_I \frac{\hbar^2}{2M_I} \frac{\nabla_I^2 A_k}{A_k}$$
(26)
$$\frac{\partial A_k}{\partial t} + \sum_I \frac{1}{M_I} \nabla_I A_k \nabla_I S_k + \sum_I \frac{1}{2M_I} A_k \nabla^2 S_k = 0$$
(27)

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If we consider the equation for the phase:

$$\frac{\partial S_k}{\partial t} + \sum_{I} \frac{1}{2M_I} (\nabla S_k)^2 + E_k = -\sum_{I} \frac{\hbar^2}{2M_I} \frac{\nabla_I^2 A_k}{A_k}$$
(28)

it is possible to take the classical limit $\hbar \rightarrow 0$ which gives the equation:

$$\frac{\partial S_k}{\partial t} + \sum_{I} \frac{1}{2M_I} \left(\nabla S_k\right)^2 + E_k = 0$$
⁽²⁹⁾

which is isomorphic to Hamilton-Jacobi of classical mechanics:

$$\frac{\partial S_k}{\partial t} + H_k(\{\mathbf{R}_I\}, \{\nabla_I S_k\}) = 0$$
(30)

with the classical Hamiltonian function

$$H_k(\{\mathbf{R}_I\}, \{\mathbf{P}_I\}) = T(\{\mathbf{P}_I\}) + V_k(\{\mathbf{R}_I\})$$
(31)

With the connecting transformation: $\mathbf{P}_I = \nabla_I S_k$.

► Route 2:

1) Maintain the quantum-mechanical time evolution for the electrons introducing the separation for the electronic and nuclear wf function in a time-dependent way

2) Time-dependent self consistent field (TDSCF) approach is obtained.

3) Ehrenfest dynamics (and as special case Born-Oppenheimer dynamics)

4) The classical limit leads then to the classical molecular dynamics for the nuclei.

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It is possible to follow an alternative route in order to maintain the dynamics of the electron.

$$\Phi(\{\mathbf{r}_i\}, \{\mathbf{R}_I\}, t) \approx \Psi(\{\mathbf{r}_i\}, t)\chi(\{\mathbf{R}_I\}, t)exp[\frac{i}{\hbar}\int_{t_0}^t \tilde{E}_e(t')dt'] \quad (32)$$

where:

$$\tilde{E}_{e} = \int \Psi^{*}(\{\mathbf{r}_{i}\}, t)\chi^{*}(\{\mathbf{R}_{I}\}, t)H_{e}\Psi(\{\mathbf{r}_{i}\}, t)\chi(\{\mathbf{R}_{I}\}, t)d\mathbf{r}d\mathbf{R}$$
(33)

Inserting this separation ansatz into $i\hbar \frac{\partial}{\partial t} \Phi(\{\mathbf{r}_i\}, \{\mathbf{R}_i\}; t) = H\Phi(\{\mathbf{r}_i\}, \{\mathbf{R}_i\}; t)$ and multiplying from the left by Ψ^* and by χ^* we obtain

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$$i\hbar\frac{\partial\Psi}{\partial t} = -\sum_{i}\frac{\hbar^{2}}{2m_{e}}\nabla_{i}^{2}\Psi \qquad (34)$$

$$+\{\int \chi^{*}(\{\mathbf{R}_{I}\},t)V_{n}e(\{\mathbf{r}_{i}\},\{\mathbf{R}_{I}\})\chi(\{\mathbf{R}_{I}\},t)d\mathbf{R}\}\Psi$$
(35)

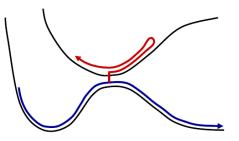
$$i\hbar\frac{\partial\chi}{\partial t} = -\sum_{I}\frac{\hbar^{2}}{2M_{I}}\nabla_{I}^{2}\chi \qquad (36)$$

$$+\{\int \Psi^*(\{\mathbf{r}_i\},t)H_e(\{\mathbf{r}_i\},\{\mathbf{R}_i\})\Psi(\{\mathbf{r}_i\},t)d\mathbf{r}\}\chi$$
(37)

This set of time-dependent Schrodinger equations define the basis of time-dependent self-consistent field (TDSCF) method. Both electrons and nuclei moves quantum-mechanically in time-dependent effective potentials.

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- Classical motion induces electronic transitions
- Quantum state determines classical forces
 - → Quantum Classical Feedback: Self-Consistency

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Using the same trick as before of writing

$$\chi_k(\lbrace \mathbf{R}_i \rbrace, t) = A_k(\lbrace \mathbf{R}_i \rbrace, t) \exp[iS_k(\lbrace \mathbf{R}_i \rbrace, t)/\hbar]$$
(38)

we obtain

$$\frac{\partial S_k}{\partial t} + \sum_{I} \frac{1}{2M_I} (\nabla S_k)^2 + \int \Psi^* H_e \Psi d\mathbf{r} = -\sum_{I} \frac{\hbar^2}{2M_I} \frac{\nabla_I^2 A_k}{A_k} \quad (39)$$
$$\frac{\partial A_k}{\partial t} + \sum_{I} \frac{1}{M_I} \nabla_I A_k \nabla_I S_k + \sum_{I} \frac{1}{2M_I} A_k \nabla^2 S_k = 0 \quad (40)$$

And in the classical limit $\hbar \to \infty$

$$\frac{\partial S_k}{\partial t} + \sum_I \frac{1}{2M_I} (\nabla S_k)^2 + \int \Psi^* H_e \Psi d\mathbf{r} = 0.$$
 (41)

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The nuclei move according to *classical* mechanics in an effective potential (Ehrenfest potential) given by the quantum dynamics of the electrons obtained by solving the time-dependent Schrodinger equation for the electrons.

$$i\hbar\frac{\partial\Psi}{\partial t}=-\sum_{i}\frac{\hbar^{2}}{2m_{e}}\nabla_{i}^{2}\Psi$$

+{
$$\int \chi^*({\mathbf{R}}_I, t) V_{ne}({\mathbf{r}}_i, {\mathbf{R}}_i) \chi({\mathbf{R}}_I, t) d{\mathbf{R}} \Psi$$
 (42)

Note: the equation (41) still contains the full quantum-mechanics nuclear wavefunction $\chi(\{\mathbf{R}_I\}, t)$. The classical reduction is obtained by:

$$\int \chi^*(\{\mathbf{R}_I\}, t) \mathbf{R}_I \chi(\{\mathbf{R}_I\}, t) d\mathbf{R} \to \mathbf{R}_I(t)$$
(43)

for $\hbar \to 0$.

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The classical limits leads to a time-dependent wave equation for the electrons

$$i\hbar\frac{\partial\Psi}{\partial t} = -\sum_{i}\frac{\hbar^{2}}{2m_{e}}\nabla_{i}^{2}\Psi + V_{ne}(\{\mathbf{r}_{i}\},\{\mathbf{R}_{I}\})\Psi \qquad (44)$$
$$i\hbar\frac{\partial\Psi}{\partial t} = H_{e}(\{\mathbf{r}_{i}\},\{\mathbf{R}_{I}\})\Psi(\{\mathbf{r}_{i}\},\{\mathbf{R}_{I}\}) \qquad (45)$$

- Feedback between the classical and quantum degrees of freedom is incorporated in both direction, even though in a mean field sense.
- These equations are called Ehrenfest dynamics in honor to Paul Ehrenfest who was the first to address the problem of how Newtonian classical dynamics of point particles can be derived from Schrodinger time-dependent wave equation.

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Difference between Ehrenfest dynamics and Born-Oppenheimer molecular dynamics:

- In ED the electronic subsystem evolves explicitly in time, according to a time-dependent Schrodinger equation
- In ED transition between electronic states are possible. This can be showed expressing the electronic wavefunction in a basis of electronic states

$$\Psi(\{\mathbf{r}_i\}, \{\mathbf{R}_I\}, t) = \sum_{l=0}^{\infty} c_l(t) \Psi_l(\{\mathbf{r}_i\}, \{\mathbf{R}_I\}, t)$$
(46)

where

$$\sum_{l=0}^{\infty} |c_l(t)|^2 = 1$$
 (47)

and one possible choice for the basis functions $\{\Psi_k\}$ is obtained solving the time-independent Schrodinger equation:

.

$$H_{e}(\{\mathbf{r}_{i}\},\{\mathbf{R}_{I}\})\Psi_{k}(\{\mathbf{r}_{i}\},\{\mathbf{R}_{I}\}) = E_{k}(\{\mathbf{R}_{I}\})\Psi_{k}(\{\mathbf{r}_{i}\},\{\mathbf{R}_{I}\}).$$
(48)



The Ehrenfest dynamics reduces to the Born-Oppenheimer molecular dynamics if only one term is considered in the sum:

$$\Psi(\{\mathbf{r}_i\},\{\mathbf{R}_I\},t)=\sum_{l=0}^{\infty}c_l(t)\Psi_l(\{\mathbf{r}_i\},\{\mathbf{R}_I\},t)$$

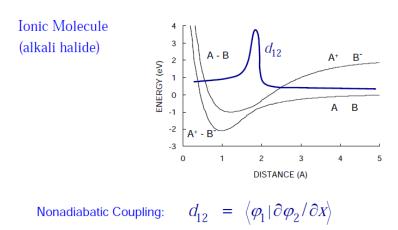
namely:

 $\Psi({\mathbf{r}_i}, {\mathbf{R}_l}, t) = \Psi_0$ ground state adiabatic wavefunction (49)

This should be a good approximation if the energy difference between Ψ_0 and the first excited state Ψ_1 is large everywhere compared to the thermal energy scale $K_B T$. In this approximation the nuclei move on a single adiabatic potential energy surface, $E_0(\{\mathbf{R}_I\})$.

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Classical trajectory calculations on global potential energy surfaces

In BO one can think to fully decouple the task of generating classical nuclear dynamics from the task of computing the quantum potential energy surface.

- ► *E*₀ is computed for many different {**R**_{*I*}}
- data points fitted to analytical function
- Newton equation of motion solved on the computed surfaces for different initial conditions

Problem: dimensionality bottleneck. It has been used for scattering and chemical reactions of small systems in vacuum, but is not doable when nuclear degrees of freedom increase.

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Force Field - based molecular dynamics

One possible solution to the dimensionality bottleneck is the force field based MD.

$$V_{e}^{E} \approx V_{e}^{FF} = \sum_{I=1}^{N} v_{1}(\mathbf{R}_{I}) + \sum_{I < J}^{N} v_{2}(\mathbf{R}_{I}, \mathbf{R}_{J}) + (50)$$
$$\sum_{I < I < K}^{N} v_{3}(\mathbf{R}_{I}, \mathbf{R}_{J}, \mathbf{R}_{K}) + \dots (51)$$

The equation of motion for the nuclei are:

$$M_I \ddot{\mathbf{R}}_I(t) = -\nabla_I V_e^{FF}(\{\mathbf{R}_I(t)\}).$$
(52)

The electrons follow adiabatically the classical nuclear motion and can be integrated out. The nuclei evolve on a single BO potential energy surface, approximated by a few body interactions.

Route 1 Route 2

Ehrenfest molecular dynamics

To avoid the dimensionality bottleneck the coupled equations:

$$M_{I}\ddot{\mathbf{R}}_{I}(t) = -\nabla_{I}\langle H_{e}\rangle \quad (53)$$
$$i\hbar\frac{\partial\Psi}{\partial t} = \left[-\sum_{i}\frac{\hbar^{2}}{2m_{e}}\nabla_{i}^{2} + V_{ne}(\{\mathbf{r}_{i}\},\{\mathbf{R}_{I}\})\chi(\{\mathbf{R}_{I}\},t)\right]\Psi \quad (54)$$

can be solved simultaneously.

The time-dependent Schrodinger equation is solved *on the fly* as the nuclei are propagated using classical mechanics.

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Using

$$\Psi(\{\mathbf{r}_{i}\},\{\mathbf{R}_{I}\},t) = \sum_{l=0}^{\infty} c_{l}(t)\Psi_{l}(\{\mathbf{r}_{i}\},\{\mathbf{R}_{I}\})$$

the Ehrenfest equations reads:

$$M_{I}\ddot{\mathbf{R}}_{I}(t) = -\nabla_{I}\sum_{k} |c_{k}(t)|^{2}E_{k}$$
(55)
$$= -\sum_{k} |c_{k}(t)|^{2}\nabla_{I}E_{k} + \sum_{k,l} c_{k}^{*}c_{l}(E_{k} - E_{l})\mathbf{d}_{l}^{kl}$$
(56)
$$i\hbar\dot{c}_{k}(t) = c_{k}(t)E_{k} - i\hbar\sum_{l} c_{k}(t)D^{kl}$$
(57)

where the non-adiabatic coupling elements are given by

$$D^{kl} = \int \Psi_k^* \frac{\partial}{\partial t} \Psi_l d\mathbf{r} = \sum_l \dot{\mathbf{R}}_l \int \Psi_k^* \nabla_l \Psi_l = \sum_l \dot{\mathbf{R}}_l \mathbf{d}^{kl}.$$
 (58)

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Car-Parrinello molecular dynamics



Description

Roberto Car with Michele Parrinello, during the Stig Lundqvist Research Conference on the Advancing Frontiers in Condensed Matter Physics, Adriatico Guest House, July 2001

- Combine the advantages of Ehrenfest and BO dynamics
- Integrate the equation of motion on a longer time-step than in Ehrenfest, but at the same time
- take advantage of the smooth time evolution of the dynamically evolving electronic subsystem

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Unified Approach for Molecular Dynamics and Density-Functional Theory

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We present a unified scheme that, by combining molecular dynamics and density-functional theory, profoundly extends the range of both concepts. Our approach extends molecular dynamics beyond the usual pair-potential approximation, thereby making possible the simulation of both covalently bonded and metallic systems. In addition it permits the application of density-functional theory to much larger systems than previously feasible. The new technique is demonstrated by the calculation of some static and dynamic properties of crystalline silicon within a self-consistent pseudopotential framework.

Car-Parrinello method just turned 25!

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Car-Parrinello Lagrangian and equations of motion

CP dynamics is based on the adiabatic separation between fast electronic (quantum) and slow (classical) nuclear motion. They introduced the following Lagrangian

$$\mathcal{L}_{CP} = \sum_{I} \frac{1}{2} M_{I} \dot{\mathbf{R}}_{I}^{2} + \sum_{i} \mu < \dot{\phi}_{i} |\dot{\phi}_{i} > - < \Psi_{0}| H_{e} |\Psi_{0} > + constraints$$
(59)

where $\Psi_0 = 1/\sqrt{N!} det\{\phi_i\}$ and μ is the fictious mass of the electrons.

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the associated Euler-Lagrange equations are:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{R}}_{I}} = \frac{\partial \mathcal{L}}{\partial \mathbf{R}_{I}}$$
(60)
$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}_{i}^{*}} = \frac{\partial \mathcal{L}}{\partial \phi_{i}^{*}}$$
(61)

from which the Car-Parrinello equations of motion:

$$M_{I}\ddot{\mathsf{R}}_{I}(t) = -\frac{\partial}{\partial \mathsf{R}_{I}} < \Psi_{0}|H_{e}|\Psi_{0} > +\frac{\partial}{\partial \mathsf{R}_{I}} \{\text{constraints}\}$$
(62)
$$\mu \ddot{\phi}_{i}(t) = -\frac{\delta}{\delta \phi_{i}^{*}} < \Psi_{0}|H_{e}|\Psi_{0} > +\frac{\delta}{\delta \phi_{i}^{*}} \{\text{constraints}\}$$
(63)

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For the specific case of the Kohn-Sham Theory the CP Lagrangian is:

$$\mathcal{L}_{CP} = \sum_{I} \frac{1}{2} M_{I} \dot{\mathbf{R}}_{I}^{2} + \sum_{i} \mu < \dot{\phi}_{i} |\dot{\phi}_{i} > - < \Psi_{0}| H_{e}^{KS} |\Psi_{0} >$$
$$+ \sum_{i,j} \Lambda_{ij} (<\phi_{i} |\phi_{j} > -\delta_{ij}) \qquad (64)$$

and the Car-Parrinello equations of motion:

$$M_{I}\ddot{\mathbf{R}}_{I}(t) = -\frac{\partial}{\partial \mathbf{R}_{I}} < \Psi_{0}|H_{e}^{KS}|\Psi_{0}>$$
(65)

$$\mu \ddot{\phi}_i(t) = -H_e^{KS} \phi_i + \sum_j \Lambda_{ij} \phi_j$$
(66)

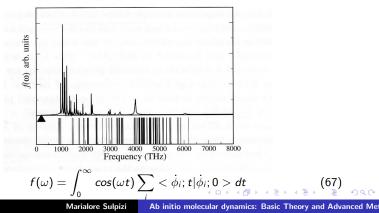
The nuclei evolve in time with temperature $\propto \sum_{I} M_{I} \dot{\mathbf{R}}_{I}^{2}$; the electrons have a fictitious temperature $\propto \sum_{i} \mu < \dot{\phi}_{i} | \dot{\phi}_{i} >$.

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Why does the CP method work?

- Separate in practice nuclear and ionic motion so that
- ► electrons keep cold, remaining close to min_{{φi}} < Ψ₀|H_e|Ψ₀ >, namely close to the Born-Oppenheimer surface

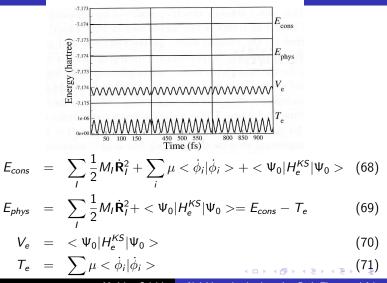


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Energy Conservation



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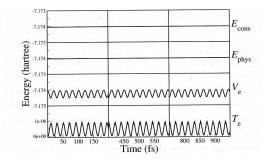
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Energy Conservation

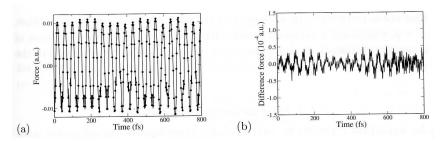


- Electrons do not heat-up, but fluctuate with same frequency as V_e
- Nuclei drag the electrons
- *E_{phys}* is essentially constant

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Deviation from Born-Oppenheimer surface



Deviation of forces in CP dynamics from the true BO forces small and/but oscillating.

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How to control of adiabaticity?

In a simple harmonic analysis of the frequency spectrum yields

$$\omega_{ij} = \sqrt{\frac{2(\epsilon_i - \epsilon_j)}{\mu}} \tag{72}$$

where ϵ_i and ϵ_j are the eigenvalues of the occupied/unoccupied orbitals of the Kohn-Sham Hamiltonian. The lowest possible electronic frequency is:

$$\omega_e^{min} \propto \sqrt{\frac{E_{gap}}{\mu}}.$$
 (73)

The highest frequency

$$\omega_e^{max} \propto \sqrt{\frac{E_{cut}}{\mu}}.$$
 (74)

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$$\Delta t^{max} \propto \sqrt{\frac{\mu}{E_{cut}}}.$$

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In order to guarantee adiabatic separation between electrons and nuclei we should have

• large $\omega_e^{min} - \omega_n^{max}$.

 ω_n^{max} and E_{gap} depend on the physical system, so the parameter to control adiabaticity is the mass μ .

However the mass cannot be reduced arbitrarily otherwise the timestep becomes too small.

Alternatively if Δt is fixed and μ is chosen

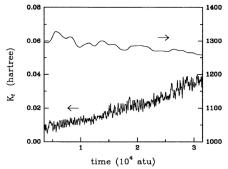
- μ too small: Electrons too light and adiabacity will be lost
- ▶ µ too large: Time step eventually large and electronic degrees of freedom evolve too fast for the Verlet algorithm

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Loss of adiabaticity: the bad cases

• Vacancy in hot 64-atom Si cell



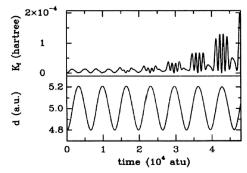
Due to the presence of the vacancy there is a small gap in the system.

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Loss of adiabaticity: the bad cases

• Sn₂: Degeneracy of HOMO and LUMO at short distances



In this system the gap is periodically opened (up to 0.3 eV) and nearly closed at short distances. The electrons gain kinetic energy in phase with the ionic oscillations.

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Zero or small electronic gaps: thermostatted electrons

- One way to (try to) overcome the problem in coupling of electronic and ionic dynamics is to thermostat also the electrons (Blöchl & Parrinello, PRB 1992)
- Thus electrons cannot heat up; if they try to, thermostat will adsorb the excess heat
- ► Target fictitious kinetic energy *E*_{kin,0} instead of temperature
- Mass of thermostat to be selected appropriately: Too light: Adiabacity violated (electrons may heat up) Too heavy: lons dragged excessively
- Note: Introducing the thermostat the conserved quantity changes

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$$M_{I}\ddot{\mathbf{R}}_{I}(t) = -\frac{\partial}{\partial \mathbf{R}_{I}} < \Psi_{0}|H_{e}^{KS}|\Psi_{0} > -M_{I}\dot{R}_{I}\dot{x}_{R}$$
(76)

$$\mu \ddot{\phi}_i(t) = -H_e^{KS} \phi_i + \sum_j \Lambda_{ij} \phi_j - \mu \dot{\phi}_i \dot{x}_e$$
(77)

in blue are the frictious terms governed by the following equations:

$$Q_{e}\ddot{x}_{e} = 2\left[\sum_{i}\mu\dot{\phi}^{2} - E_{kin,0}\right]$$

$$Q_{R}\ddot{x}_{R} = 2\left[\sum_{I}\frac{1}{2}M_{I}\dot{R}^{2} - \frac{1}{2}gK_{B}T\right]$$
(78)
(79)

The masses Q_e and Q_R determines the time scale for the thermal fluctuations. The conserved quantity is now:

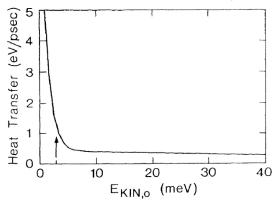
$$E_{tot} = \sum_{I} \frac{1}{2} M_{I} \dot{\mathbf{R}}_{I}^{2} + \sum_{i} \mu < \dot{\phi}_{i} |\dot{\phi}_{i} > + < \Psi_{0} | H_{e}^{KS} | \Psi_{0} >$$

$$+ \frac{1}{2} Q_{e} \dot{x}_{e}^{2} + 2E_{kin,0} x_{e} + \frac{1}{2} Q_{R} \dot{x}_{R}^{2} + g_{\Box B} T \dot{x}_{R} = (80)$$
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Thermostat on the electrons: examples

- Example: Aluminium
- Dependence of the heat transfer on the choice of $E_{kin,0}$ in solid AI



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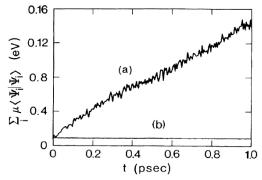
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Thermostat on the electrons: examples

- 64 atoms of molten aluminium
- (a): Without thermostat
- (b): With thermostat

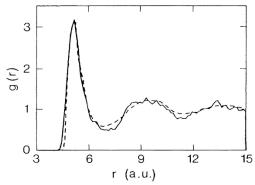


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Thermostat on the electrons: examples

- Check: Radial pair correlation function
 - Solid line: CP-MD with thermostat
 - Dashed line: Calculations by Jacucci et al



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Analytic and numerical error estimates

- The CP forces necessarily deviate from the BO
- The primary effect of μ makes the ions "heavier"
- No effect on the thermodynamical and structural properties, but affect the dynamical quantities in a *systematic* way (vibrational spectra)

$$\phi_i(t) = \phi_i^0(t) + \delta\phi_i(t) \tag{81}$$

Inserting this expression into the CP equations of motions:

$$\mathbf{F}_{I,\alpha}^{CP}(t) = \mathbf{F}_{I,\alpha}^{BO}(t) + \sum_{i} \mu\{\langle \ddot{\phi}_{i} | \frac{\partial |\phi_{i}^{0} \rangle}{\partial \mathbf{R}_{I,\alpha}} \frac{\partial \langle \phi_{i}^{0} |}{\partial \mathbf{R}_{I,\alpha}} | \ddot{\phi}_{i} \rangle\} + \mathcal{O}(\delta \phi_{i}^{2}) \quad (82)$$

the additional force is linear in the mass μ so that it vanishes properly as $\mu \to 0.$

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The new equations of motion:

$$(M_I + \Delta_\mu M_I)\ddot{R}_I = F_I \tag{83}$$

where, in the isolated atom approximation,

$$\Delta_{\mu}M_{I} = \frac{2}{3}\mu E_{kin}^{I} = \frac{2}{3}\frac{m_{e}}{\hbar^{2}}\sum_{j} <\phi_{j}^{I}| - \frac{\hbar^{2}}{2m_{e}}\nabla_{j}^{2}|\phi_{j}^{I}>>0$$
(84)

is an unphysical mass, or drag, due to the fictitious kinetics of the electrons

 for a system where electrons are strongly localized close to the nuclei there more pronounced renormalization effect

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Example: Vibrations in water molecule

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mode	harmonic	BOMD	50	100	200	400	dM/M [%]
bend	1548	1543	1539	1535	1529	1514	$0.95 \times 10^{-3} \mu$
sym.	3515	3508	3494	3478	3449	3388	$1.81{ imes}10^{-3}\mu$
asym.	3621	3616	3600	3585	3556	3498	$1.71{ imes}10^{-3}\mu$

To correct for finite- μ effects:

- ▶ Perform simulation for different μ -values and extrapolate for $\mu \rightarrow 0$.
- use mass renormalization according to:

$$\omega_{BO} = \omega_{CP} \sqrt{1 + \frac{\Delta_{\mu} M}{M}} \tag{85}$$

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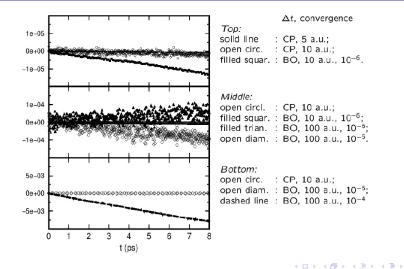
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Car-Parrinello vs Born-Oppenheimer dynamics

Born-Oppenheimer MD	Car-Parrinello MD
Exactly on BO surface	Always slightly off BO surface
$\Delta t pprox$ ionic time scales, maximum time step possible	$\Delta t \ll$ ionic time scales, (much) shorter time step necessary
maximum time step possible	(much) shorter time step necessary
Expensive minimisation	Orthogonalisation only,
at each MD step	less expensive per MD step
Not stable against deviations	Stable against deviations
from BO surface	from BO surface
\Rightarrow Energy/temperature drift,	
thermostatting of ions necessary	
Same machinery in zero-gap syst	ems Thermostatting of electrons
	to prevent energy exchange
Most applications in solids	Superior for liquids
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CP vs BO: energy conservation



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CP vs BO: timing

STABILITY OF BO AND CP MD COMPARED

CPMD results for the 8 Si atom model system

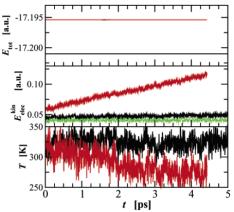
Timings in cpu seconds and energy conservation in a.u. / ps for CP and BO Molecular dynamics simulations for 1 ps of trajectory on an IBM RS6000 / model 390 (Power2) workstation

Method	Time step	Convergence	Conservation (au/ps)	⊤ime (s)
CP	5		6×10 ⁻⁸	3230
CP	7		1×10 7	2310
CP	10		3×10 ⁻⁷	1610
BO	10	10^{-6}	1×10^{-6}	16590
BO	50	10^{-6}	1×10 ⁻⁶	4130
BO	100	10^{-6}	6×10 ⁻⁶	2250
BO	100	10^{-5}	1×10^{-5}	1660
BO	100	10-4	1×10 ⁻³	1060

Timing in CPU seconds and energy conservation in a.u. Pos for the second Advanced Met Ab initio molecular dynamics: Basic Theory and Advanced Met

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CP of liquid water: Energy conservation



 CPMD-800-NVE-64

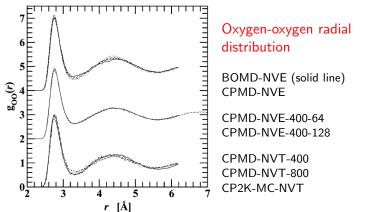
 CPMD-400-NVE-128

 CPMD-800-NVT-64
 J. Phys. Chem. B 2004, 108, 12990. こののの

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CP of liquid water: structure



The radial distribution functions are correct and independent of the method used.

J. Phys. Chem. B 2004, 108, 12990.

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Car-Parrinello method: Summary

- Molecular dynamics can be used to perform real-time dynamics in atomistic systems
- Verlet algorithm yields stable dynamics (in CPMD implemented algorithm velocity Verlet)
- Born-Oppenheimer dynamics: Max time step 1 fs (highest ionic frequency 2000-3000 cm⁻¹)
 Car-Parrinello dynamics: Max time step 0.1 fs
- Car-Parrinello method can yield very stable dynamical trajectories, provided the electrons and ions are adiabatically decoupled
- The method is best suited for e. g. liquids and large molecules with a wide electronic gap
- The speed of the method is comparable or faster than using Born-Oppenheimer dynamics and still more accurate (i. e. stable)
- ▶ One has to be careful with the choice of $\mu! < \square > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ >$

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Forces calculation

Efficient calculation of the forces One possibility is the numerical evaluation of :

$$\mathbf{F}_{I} = -\nabla \langle \Psi_{0} | H_{e} | \Psi_{0} \rangle \tag{86}$$

which can be costly and not so accurate. So analytical derivative is desiderable:

$$\nabla_I \langle \Psi_0 | H_e | \Psi_0 \rangle =$$
(87)

$$\langle \Psi_0 | \nabla_I | H_e | \Psi_0 \rangle + \langle \nabla_I \Psi_0 | H_e | \Psi_0 \rangle + \langle \Psi_0 | | H_e | \nabla_I \Psi_0 \rangle$$
(88)

For the Hellmann Feymann Theorem:

$$\mathbf{F}_{I}^{HFT} = \langle \Psi_{0} | \nabla_{I} H_{e} | \Psi_{0} \rangle \tag{89}$$

if Ψ_0 is en exact eigenfunction of H_e .

in numerical calculation that is not the case and contribution to the forces also arise from variation of the wf with respect to atomic positions.

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We need to calculate the additional forces coming form the wf contribution.

Let's start with

$$\Psi_0 = 1/N! det\phi_i \tag{90}$$

where the orbitals ϕ_i can be expanded as:

$$\phi_i = \sum_{\nu} c_{i\nu} f_{\nu}(r, \{R_l\})$$
(91)

Two contributions to the forces emerge from the variation of the wf:

$$\nabla_{I}\phi_{i} = \sum_{\nu} (\nabla_{I}c_{i\nu}f_{\nu}(r, \{R_{I}\}) + \sum_{\nu}c_{i\nu}(\nabla_{I}f_{\nu}(r, \{R_{I}\}))$$
(92)

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We calculate now the contribution due to the incomplete-basis-set correction. Using:

$$\nabla_{I}\phi_{i} = \sum_{\nu} (\nabla_{I}c_{i\nu}f_{\nu}(r, \{R_{I}\}) + \sum_{\nu}c_{i\nu}(\nabla_{I}f_{\nu}(r, \{R_{I}\}))$$
(93)

into

$$\langle \nabla_I \Psi_0 | H_e | \Psi_0 \rangle + \langle \Psi_0 | | H_e | \nabla_I \Psi_0 \rangle$$
(94)

we obtain:

$$\sum_{i\mu\nu} c_{i\mu}^* c_{i\nu} (\langle \nabla_I f_\mu | H_e | f_\nu \rangle + \langle f_\mu | H_e | \nabla_I f_\nu \rangle) +$$
(95)

$$\sum_{i\mu\nu} (\nabla_{I} c_{i\mu}^{*} H_{\mu\nu} c_{i\nu} + c_{i\mu}^{*} H_{\mu\nu} \nabla_{I} c_{i\nu})$$
(96)

Making use of

$$\sum_{\nu} H_{\mu\nu} c_{i\nu} = \epsilon_i \sum_{\nu} S_{\mu\nu} c_{i\nu}$$
(97)

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We obtain:

$$\sum_{i\mu\nu} c_{i\mu}^* c_{i\nu} (\langle \nabla_I f_{\mu} | H_e | f_{\nu} \rangle + \langle f_{\mu} | H_e | \nabla_I f_{\nu} \rangle) + \sum_{i\mu\nu} \epsilon_i (\nabla_I c_{i\mu}^* S_{\mu\nu} c_{i\nu} + c_{i\mu}^* S_{\mu\nu} \nabla_I c_{i\nu})$$
(98)

which can be rewritten as:

$$\sum_{i\mu\nu} c_{i\mu}^* c_{i\nu} (\langle \nabla_I f_\mu | H_e | f_\nu \rangle + \langle f_\mu | H_e | \nabla_I f_\nu \rangle) + \sum_{i\mu\nu} \epsilon_i S_{\mu\nu} \nabla_I (c_{i\mu}^* c_{i\nu})$$
(99)

and

$$\sum_{i\mu\nu} c_{i\mu}^* c_{i\nu} (\langle \nabla_I f_\mu | H_e | f_\nu \rangle + \langle f_\mu | H_e | \nabla_I f_\nu \rangle) +$$
(100)

$$\sum_{i} \epsilon_{i} \left[\nabla_{I} \sum_{\mu\nu} S_{\mu\nu} c_{i\mu}^{*} c_{i\nu} - \sum_{i\mu\nu} \nabla_{I} S_{\mu\nu} c_{i\mu}^{*} c_{i\nu} \right]$$
(101)

And finally:

$$\sum_{i,\dots} c_{i\mu}^* c_{i\nu} (\langle \nabla_I f_{\mu} | H_e - \epsilon_i | f_{\nu} \rangle + \langle f_{\mu} | H_e_{\epsilon_{i}} - \epsilon_i | \nabla_I f_{\nu} \rangle)_{\epsilon_{i}} + \epsilon_{i} | \nabla_I f_{\nu} \rangle)_{\epsilon_{i}}$$

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In the case of plane waves Pulay forces vanish

- this is due to the fact that plane wave do not depend on the atomic coordinates

- it is only true if number of plane waves is kept fixed. An additional contribution to the forces comes from the non-self-consistency

$$\mathbf{F}^{\mathsf{NSC}}{}_{I} = -\int dr (\nabla_{I} n) (V^{SCF} - V^{NSC})$$
(103)

Such a force vanishes when the self consistency is reached, meaning Ψ_0 is the exact wavefunction within the subspace spanned by the finite basis set.

In Car-Parrinello, as well as in Ehrenfest MD there is no minimization full self-consistency is not required, so this force is irrelevant.

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What electronic structure method?

CP MD is not tied to any particular electronic structure approach.

DFT is very often the choice

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The Hartree Fock Approximation

$$\Psi_{HF} = \frac{1}{\sqrt{N!}} \begin{vmatrix} \psi_1(x_1) & \psi_2(x_1) & \dots & \psi_N(x_1) \\ \psi_1(x_2) & \psi_2(x_2) & \dots & \psi_N(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \psi_1(x_N) & \psi_2(x_N) & \dots & \psi_N(x_N) \end{vmatrix} = \frac{1}{\sqrt{N!}} det[\psi_1 \psi_2 \cdots \psi_N]$$
(104)
orthogonal orbitals ψ_i are found minimizing $E[\Psi] = \frac{\langle \Psi | \hat{H} | \Psi \rangle}{\langle \Psi | \Psi \rangle}$ where Ψ is in the form of a Slater determinant. $\langle \Psi_{HF} | \Psi_{HF} \rangle = 1$

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The Hartree Fock Approximation

$$E_{HF} = \langle \Psi_{HF} | \hat{H} | \Psi_{HF} \rangle = \sum_{i=1}^{N} H_i + \frac{1}{2} \sum_{i,j=1}^{N} (J_{ij} - K_{ij})$$
(105)

where

$$H_{i} = \int \psi_{i}^{*}(x) [-\frac{1}{2}\nabla^{2} + v(x)]\psi_{i}(x)dx \qquad (106)$$

$$J_{ij} = \iint \psi_i(x_1)\psi_i^*(x_1)\frac{1}{r_{12}}\psi_j^*(x_2)\psi_j(x_2)dx_1dx_2$$
(107)

$$K_{ij} = \iint \psi_i^*(x_1)\psi_j(x_1)\frac{1}{r_{12}}\psi_i(x_2)\psi_j^*(x_2)dx_1dx_2$$
(108)

$$J_{ii} = K_{ii} \tag{109}$$

this explain the double sum in (105).

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The Hartree Fock Approximation

Minimizing E_{HF}

$$E_{HF} = \langle \Psi_{HF} | \hat{H} | \Psi_{HF} \rangle = \sum_{i=1}^{N} H_i + \frac{1}{2} \sum_{i,j=1}^{N} (J_{ij} - K_{ij})$$
(110)

subject to the orthonormalization conditions

$$\int \psi_i^*(x)\psi_j(x)dx = \delta_{ij} \tag{111}$$

gives the Hartree Fock differential equations:

$$\hat{F}\psi_i(x) = \sum_{j=1}^N \epsilon_{ij}\psi_j(x) \tag{112}$$

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$$\hat{F}\psi_i(x) = \sum_{k=1}^N \epsilon_{ij}\psi_j(x)$$
 where

$$\hat{F} = -\frac{1}{2}\nabla^2 + v(x) + \hat{g}$$
 (113)

in which the Coulomb-exchange operator \hat{g} is given by

$$\hat{g} = \hat{j} - \hat{k}. \tag{114}$$

Here

$$\hat{f}(x_1)f(x_1) \equiv \sum_{k=1}^N \int \psi_k^*(x_2)\psi_k(x_2)\frac{1}{r_{12}}f(x_1)dx_2 \qquad (115)$$

and

$$\hat{k}(x_1)f(x_1) \equiv \sum_{k=1}^{N} \int \psi_k^*(x_2)f(x_2) \frac{1}{r_{12}} \psi_k(x_1) dx_2$$
(116)

where $f(x_1)$ is an arbitrary function and the matrix ϵ consists of Lagrange multipliers.

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Density Functional Theory

Let's consider a *noninteracting reference system* in which there is no electron electron repulsion term and for which the electron density is exactly ρ :

$$\hat{H}_{s} = \sum_{i}^{N} \left(-\frac{1}{2} \nabla_{i}^{2} \right) + \sum_{i}^{N} v_{s} \left(r_{i} \right)$$
(117)

For this system we have an exact solution

$$\Psi_s = \frac{1}{\sqrt{N!}} \det \left[\psi_1 \psi_2 ... \psi_N \right] \tag{118}$$

for such a system the kinetic energy is

$$T_{s}[\rho] = \langle \Psi_{s}| \sum_{i}^{N} \left(-\frac{1}{2}\nabla_{i}^{2}\right) |\Psi_{s}\rangle =$$
(119)

$$= \sum_{i}^{N} \langle \Psi_{s} | -\frac{1}{2} \nabla_{i}^{2} | \Psi_{s} \rangle \qquad (120)$$

Ab initio molecular dynamics: Basic Theory and Advanced Met

Marialore Sulpizi

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Now the very clever idea by Kohn and Sham was to rewrite

$$E_{v}\left[
ho
ight]=T\left[
ho
ight]+V_{ne}\left[
ho
ight]+V_{ee}\left[
ho
ight]$$

as

$$E_{v}[\rho] = T_{s}[\rho] + V_{ne}[\rho] + J[\rho] + E_{xc}[\rho]$$
(121)

where

$$E_{xc}[\rho] = T[\rho] - T_{s}[\rho] + V_{ee}[\rho] - J[\rho]$$
(122)

The quantity $E_{xc}[\rho]$ is the exchange-correlation energy , which contains the difference between $T[\rho]$ and $T_s[\rho]$ and the non-classical part of $V_{ee}[\rho]$.

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$$E\left[\rho\right] = T_{s}\left[\rho\right] + J\left[\rho\right] + E_{xc}\left[\rho\right] + \int v(r)\rho(r)dr =$$
$$= \sum_{i}^{N} \sum_{s} \int \psi_{i}^{*}(x_{i}) \left(-\frac{1}{2}\nabla^{2}\right) \psi_{i}(x_{i})dr + J\left[\rho\right] + E_{xc}\left[\rho\right] + \int v(r)\rho(r)dr$$

and the electron density

$$\rho(\mathbf{r}) = \sum_{i}^{N} \sum_{s} |\psi_i(\mathbf{r}, s)|^2$$
(123)

The variational search for the minimum of $E[\rho]$ can be equivalently effected in the space of orbitals ψ_i , constraining the orbitals to be orthonormal:

$$\int \psi_i^*(x)\psi_j(x)dx = \delta_{ij}.$$
(124)

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If we now define the functional

$$\Omega[\psi_i] = E[\rho] - \sum_i^N \sum_j^N \epsilon_{ij} \int \psi_i^*(x) \psi_j(x) dx \qquad (125)$$

where ϵ_{ij} are the Lagrange multiplier for the orthonormality constraints, For $E[\rho]$ to be a minimum it is necessary that:

$$\delta\Omega\left[\psi_i\right] = 0,\tag{126}$$

which leads to the equations:

$$\hat{h}_{eff}\psi_i = \left[-\frac{1}{2}\nabla^2 + v_{eff}\right]\psi_i = \sum_j^N \epsilon_{ij}\psi_j$$
(127)

where

$$v_{eff}(r) = v(r) + \int \frac{\rho(r')}{|r - r'|} dr' + v_{xc}(r).$$
(128)

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 ϵ_{ij} is Hermitian and can be diagonalized by unitary transformation. Such a transformation leaves invariant the Slater determinant, the density and hence the Hamiltonian.

In this way we can obtain the *Kohn-Sham* equations in their canonical form:

$$\left[-\frac{1}{2}\nabla^2 + \mathbf{v}_{eff}\right]\psi_i = \epsilon_i\psi_i \tag{129}$$

$$v_{eff}(r) = v(r) + \int \frac{\rho(r')}{|r-r'|} dr' + v_{xc}(r)$$
 (130)

$$\rho(r) = \sum_{i}^{N} \sum_{s} |\psi_{i}(r, s)|^{2}$$
(131)

These equations are non-linear and must be solved iteratively.

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Then the total energy can be calculated from the energy functional:

$$E[\rho] = T_s[\rho] + J[\rho] + E_{xc}[\rho] + \int v(r)\rho(r)dr$$

or in terms of the K-S eigenvalues by:

$$E = \sum_{i}^{N} \epsilon_{i} - \int v_{eff}(r)\rho(r)dr + J[\rho] + E_{xc}[\rho] + \int v(r)\rho(r)dr$$
$$E = \sum_{i}^{N} \epsilon_{i} - \frac{1}{2}\int \frac{\rho(r)\rho(r')}{|r-r'|}drdr' + E_{xc}[\rho] - \int v_{xc}(r)\rho(r)(132)$$

Here

$$\sum_{i}^{N} \epsilon_{i} = \sum_{i}^{N} \langle \psi_{i} | -\frac{1}{2} \nabla^{2} + v_{eff}(r) | \psi_{i} \rangle$$
(133)

$$= T_{s}[\rho] + \int v_{eff}(r)\rho(r)dr \qquad (134)$$

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Local-density

An explicit form of $E_{xc}[\rho]$ is needed!

$$E_{xc}^{LDA}[\rho] = \int \rho(r)\epsilon_{xc}(\rho)dr \qquad (135)$$

where $\epsilon_{xc}(\rho)$ is the exchange and correlation energy per particle of a uniform electron gas of density ρ . the corresponding xc potential is

$$v_{xc}^{LDA}(r) = \frac{\delta E_{xc}^{LDA}}{\delta \rho(r)} = \epsilon_{xc}(\rho(r)) + \rho(r) \frac{\partial \epsilon_{xc}(\rho)}{\partial \rho}$$
(136)

and the KS orbitals read:

$$\left[-\frac{1}{2}\nabla^2 + v(r) + \int \frac{\rho(r')}{|r-r'|} dr' + v_{xc}^{LDA}(r)\right]\psi_i = \epsilon_i\psi_i \qquad (137)$$

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Gradient expansion

Among limitations of LDA there the fact that inhomogeneities in the density are not taken into account. The can be introduced semi-locally by expanding $E_{xc} [\rho]$ in term of density and its gradient.

$$E_{xc}\left[\rho\right] = \int \rho(\mathbf{r})\epsilon_{xc}\left[\rho(\mathbf{r})\right]F_{xc}\left[\rho(\mathbf{r}),\nabla\rho(\mathbf{r}),\nabla^{2}\rho(\mathbf{r}),\ldots\right]d\mathbf{r} \quad (138)$$

the second order gradient expansion corresponds to:

$$E_{xc}[\rho] = \int A_{xc}[\rho] \rho(\mathbf{r})^{4/3} d\mathbf{r} + \int C_{xc}[\rho] |\nabla[\rho]|^2 / \rho(\mathbf{r})^{4/3} d\mathbf{r}$$
(139)

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Gradient expansion

GGAs have been obtained following two different approaches:

- Theoretical derivation of an appropriate expression with coefficients chosen so that number of formal conditions are fulfilled: namely sum rules, long range decay etc...
- Fit the parameters of the functional in order to reproduce a number of experimental results in a molecular database (structural parameters, formation energies, thermochemical data (possible issue: transferability).

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