Antworten zu den Beispielaufgaben zum Thema **Partielle Ableitung**

Aufgabe 6.1 Berechnen Sie die ersten partiellen Ableitungen der Funktion f:

a) $f(x,y) = x^3 + y^3 - 3xy$,

$$\frac{\partial f}{\partial x} = 3x^2 - 3y, \quad \frac{\partial f}{\partial y} = 3y^2 - 3x.$$

b) $f(x,y) = \dfrac{x(x-y)}{y^2}$,

$$\frac{\partial f}{\partial x} = \frac{2x-y}{y^2}, \quad \frac{\partial f}{\partial y} = \frac{xy-2x^2}{y^3}.$$

c) $f(x,y) = \sin\dfrac{x}{y}\cos\dfrac{y}{x}$,

$$\frac{\partial f}{\partial x} = \frac{1}{y}\cos\frac{x}{y}\cos\frac{y}{x} + \frac{y}{x^2}\sin\frac{x}{y}\sin\frac{y}{x}, \quad \frac{\partial f}{\partial y} = -\frac{x}{y^2}\cos\frac{x}{y}\cos\frac{y}{x} - \frac{1}{x}\sin\frac{x}{y}\sin\frac{y}{x}.$$

d) $f(x,y) = e^x(\cos y + x\sin y)$,

$$\frac{\partial f}{\partial x} = e^x(x\sin y + \sin y + \cos y), \quad \frac{\partial f}{\partial y} = e^x(x\cos y - \sin y).$$

e) $f(x,y,z) = xy + yz + zx$,

$$\frac{\partial f}{\partial x} = y + z, \quad \frac{\partial f}{\partial y} = z + x, \quad \frac{\partial f}{\partial z} = x + y.$$

f) $f(x,y,z) = \dfrac{x}{z} + \dfrac{z}{x}$,

$$\frac{\partial f}{\partial x} = \frac{1}{z} - \frac{z}{x^2}, \quad \frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial z} = \frac{1}{x} - \frac{x}{z^2}.$$

g) $f(x,y,z) = \dfrac{y}{z} + \arctan\dfrac{z}{x} + \arctan\dfrac{x}{z}$,

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = \frac{1}{z}, \quad \frac{\partial f}{\partial z} = -\frac{y}{z^2}.$$

h) $f(x,y,z) = z^{xy}$,

$$\frac{\partial f}{\partial x} = yz^{xy}\ln z, \quad \frac{\partial f}{\partial y} = xz^{xy}\ln z, \quad \frac{\partial f}{\partial z} = xyz^{xy-1}.$$

Aufgabe 6.2 Berechnen Sie den Gradienten der Funktion f im gegebenen Punkt v:

a) $f(x,y) = \dfrac{x}{y^2}$, $v = (1,1)$. $\quad \operatorname{grad} f|_{(1,1)} = (1,-2)$.

b) $f(x,y) = \ln\left(1 + \dfrac{x}{y}\right)$, $v = (1,2)$. $\quad \operatorname{grad} f|_{(1,2)} = \left(\frac{1}{3}, -\frac{1}{6}\right)$.

c) $f(x,y) = xye^{\sin(\pi xy)}$, $v = (1,1)$. $\quad \operatorname{grad} f|_{(1,1)} = (1-\pi, 1-\pi)$.

d) $f(x,y,z) = \dfrac{x}{x^2+y^2+z^2}$, $v = (1,0,1)$. $\quad \operatorname{grad} f|_{(1,0,1)} = \big(0, 0, -\frac{1}{2}\big)$.

e) $f(x,y,z) = \arctan \dfrac{xy}{z^2}$, $v = (3,2,1)$. $\quad \operatorname{grad} f|_{(3,2,1)} = \big(\frac{2}{37}, \frac{3}{37}, -\frac{12}{37}\big)$.

f) $f(x,y,z) = \left(xy + \dfrac{x}{y}\right)^z$, $v = (1,1,1)$. $\quad \operatorname{grad} f|_{(1,1)} = (2, 0, \ln 4)$.

Aufgabe 6.3 Berechnen Sie die zweiten partiellen Ableitungen der Funktion f:

a) $f(x,y) = xy(x^3+y^3-3)$,

$$\frac{\partial^2 f}{\partial x^2} = 12x^2y, \quad \frac{\partial^2 f}{\partial x \partial y} = 4(x^3+y^3)-3, \quad \frac{\partial^2 f}{\partial y^2} = 12xy^2.$$

b) $f(x,y) = e^{xy}$,

$$\frac{\partial^2 f}{\partial x^2} = y^2e^{xy}, \quad \frac{\partial^2 f}{\partial x \partial y} = (1+xy)e^{xy}, \quad \frac{\partial^2 f}{\partial y^2} = x^2e^{xy}.$$

c) $f(x,y) = \arctan \dfrac{x+y}{1-xy}$, $xy \neq 1$,

$$\frac{\partial^2 f}{\partial x^2} = -\frac{2x}{(1+x^2)^2}, \quad \frac{\partial^2 f}{\partial x \partial y} = 0, \quad \frac{\partial^2 f}{\partial y^2} = -\frac{2y}{(1+y^2)^2}.$$

d) $f(x,y,z) = x(1+y^2z^3)$,

$$\frac{\partial^2 f}{\partial x^2} = 0, \quad \frac{\partial^2 f}{\partial x \partial y} = 2yz^3, \quad \frac{\partial^2 f}{\partial x \partial z} = 3y^2z^2, \quad \frac{\partial^2 f}{\partial y^2} = 2xz^3, \quad \frac{\partial^2 f}{\partial y \partial z} = 6xyz^2, \quad \frac{\partial^2 f}{\partial z^2} = 6xy^2z.$$

e) $f(x,y,z) = \sin(x+y+z)$,

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial x \partial z} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z^2} = -\sin(x+y+z).$$

Aufgabe 6.4 Untersuchen Sie die Funktion $u(x,y)$ auf lokale Extrema:

a) $u(x,y) = x^2+xy+y^2-12x-3y$,

$(7,-2)$ lokales Minimum, $u(7,-2) = -39$.

b) $u(x,y) = 3x+6y-x^2-xy+y^2$,

keine Extrema.

c) $u(x,y) = 3x^2y+y^3-12x-15y+3$,

$(1,2)$ lokales Minimum, $u(1,2) = -25$, $\quad (-1,-2)$ lokales Maximum, $u(-1,-2) = 31$.

d) $u(x,y) = x^4+y^4-2x^2$,

$(1,0)$ lokales Minimum, $u(1,0) = -1$, $\quad (-1,0)$ lokales Minimum, $u(-1,0) = -1$.

e) $u(x,y) = (x+y^2)e^{\frac{x}{2}}$,

$(-2,0)$ lokales Minimum, $u(-2,0) = -\dfrac{2}{e}$.