Übungen zur Stochastik II — WS 2014/2015

Blatt 4

Problem 4.1 a) Let $\mu_n = \mathcal{N}(0, n)$ be the normal distribution with mean 0 and variance n on $\mathbb{R}$, ν the 0-measure (i.e., $\nu(A) = 0$ for all $A \in \mathcal{B}(\mathbb{R})$), λ the Lebesgue measure on $\mathbb{R}$. Check that $\mu_n \xrightarrow{\text{v}} \nu$ but the sequence (μ_n) does not converge weakly; furthermore, $\sqrt{2\pi n}\mu_n \xrightarrow{\text{v}} \lambda$.

b) For $n \in \mathbb{N}$ let X_n be geometric with parameter $p_n \in (0, 1)$, i.e., $\mathbb{P}(X_n = k) = p_n(1 - p_n)^k$, $k \in \mathbb{N}_0$. Under which conditions on the sequence $(p_n)_{n\in\mathbb{N}}$ do we have $X_n/n \Rightarrow \text{Exp}(\alpha)$ for $\alpha > 0$?

c) Let $X_1, X_2, \dots$ i.i.d. Exp(1), put $M_n := \max\{X_1, \dots, X_n\}$. The *Gumbel distribution* Gu (named after Emil J. Gumbel, 1891–1966) has distribution function $y \mapsto \exp(-e^{-y}), y \in \mathbb{R}$. Check that $M_n - \log n \Rightarrow \text{Gu}$.
[*Hint:* $\mathbb{P}(M_n \leq a) = \mathbb{P}(X_1 \leq a)^n$.]

Problem 4.2 Let $P \in \mathcal{M}_1([0, \infty))$ with $m_P := \int x\, P(dx) \in (0, \infty)$, define a probability measure $\widehat{P}(A) \in \mathcal{M}_1([0, \infty))$ via

$$\widehat{P}(A) := \frac{1}{m_P} \int_A x\, P(dx), \quad A \in \mathcal{B}([0, \infty)).$$

$\widehat{P}$ is the so-called *size-biased* distribution corresponding to P.

Let $(X_i)_{i\in I}$ non-negative real random variables with $\mathbb{E}[X_i] = 1$ for all $i \in I$, $P_i := \mathscr{L}(X_i)$. Check that

$$\{\widehat{P}_i : i \in I\} \text{ tight} \quad \Longleftrightarrow \quad \{X_i : i \in I\} \text{ uniformly integrable.}$$

Problem 4.3 Show that the Stone-Weierstraß theorem does in general not hold when E is not compact.

[*Hint:* Consider $E = \mathbb{R}$, construct a countable algebra $\mathcal{C}' \subset \mathcal{C}_b(\mathbb{R})$ so that $\mathcal{C}'$ separates points in $\mathbb{R}$, then use that $\mathcal{C}_b(\mathbb{R})$ is not separable.]

Problem 4.4 Let X and Y be independent, real-valued random variables. Show that $X+Y =^d X$ implies $\mathbb{P}(Y = 0) = 1$.

[*Hint:* Consider the characteristic functions of X and of $X + Y$.]

Problem 4.5* (Moment problem) Let X be a real-valued random variable with

$$\limsup_{n\to\infty} \frac{1}{n} \big(\mathbb{E}[|X|^n]\big)^{1/n} < \infty.$$

Check that the characteristic function φ_X is analytic. Furthermore, the distribution of X is determined by its moments: If Y is a real-valued random variable with $\mathbb{E}[Y^n] = \mathbb{E}[X^n]$ for all $n \in \mathbb{N}$ then $Y =^d X$.

[*Hint:* You can use the fact that $\mathbb{E}[|X|^n] < \infty$ implies that $\varphi_X \in C^n(\mathbb{R})$ with $\frac{d^n}{dt^n}\varphi(t) = \mathbb{E}\big[(iX)^n e^{itX}\big]$.]

Please turn over

Problem 4.6* **(The Prohorov metric generates the topology of weak convergence.)** Let (E,d) be a metric space, for $B \subset F$, $\varepsilon > 0$ put $B^\varepsilon := \{y \in F : d(y,B) < \varepsilon\}$. For $\mu, \nu \in \mathcal{M}_1(E)$ define

$$d_P(\mu,\nu) := \inf\big\{\varepsilon > 0 : \mu(F) \le \nu(F^\varepsilon) + \varepsilon \ \text{ for all closed } F \subset E\big\} \ (\ge 0).$$

a) Check that d_P is a metric on $\mathcal{M}_1(E)$, i.e.,

i) $d_P(\mu,\nu) = d_P(\nu,\mu)$

ii) $d_P(\mu,\nu) = 0$ if and only if $\mu = \nu$

iii) $d_P(\nu,\nu') \le d_P(\nu,\mu) + d_P(\mu,\nu')$

b) Assume that (E,d) is a complete, separable metric space. Show that then $\mu_n \xrightarrow{\mathrm{w}} \mu \iff d_P(\mu_n,\mu) \to 0$.

[*Hints*: a) Note that $E \setminus F^\varepsilon$ is closed and $(E\setminus F^\varepsilon)^\varepsilon \subset E \setminus F$; closed subsets are a $\cap$-stable generator of $\mathcal{B}(E)$; for $\varepsilon, \delta > 0$, $\overline{F^\varepsilon}^\delta = F^{\varepsilon+\delta}$. b) Use the Portmanteau theorem, for "$\Rightarrow$" use also that a weakly convergent sequence in $\mathcal{M}_1(E)$ is tight.]

Abgabe der Aufgaben: 16.12.2014 (Es findet keine Bewertung statt.)