Übungen zur Stochastik II — WS 2014/2015

Blatt 5

Problem 5.1 A function $f : \mathbb{R}^d \to \mathbb{C}$ is called *positive semi-definite* if

$$\sum_{i,j=1}^{n} \alpha_i \overline{\alpha_j} f(t_i - t_j) \geqslant 0 \quad \text{holds for all } n \in \mathbb{N}, t_1, \dots, t_n \in \mathbb{R}^d, \alpha_1, \dots, \alpha_n \in \mathbb{C}.$$

Check that the characteristic function φ_μ of a finite measure $\mu \in \mathcal{M}_f(\mathbb{R}^d)$ is positive semi-definite.
[*Note.* Bochner's theorem shows that every continuous positive semi-definite function is the characteristic function for some measure, see, e.g., W. Feller, *An Introduction to Probability Theory*, Vol. 2, Wiley 1971, Ch. XIX.2.]

Problem 5.2 a) Let $(Y_1, \dots, Y_n)$ be an n-dimensional multivariate normal random vector, $1 \leqslant m < n$. Show that

i) $Y_1, \dots, Y_n$ are independent $\iff$ $\mathrm{Cov}(Y_i, Y_j) = 0$ for $1 \leqslant i \neq j \leqslant n$, and
ii) $(Y_1, \dots, Y_m)$ is independ. from $(Y_{m+1}, \dots, Y_n)$ $\iff$ $\mathrm{Cov}(Y_i, Y_j) = 0$ for $1 \leqslant i \leqslant m < j \leqslant n$.

b) Let $X_1, \dots, X_n$ be i.i.d. $\sim \mathcal{N}(\mu, \sigma^2)$, $\overline{M} := \frac{1}{n}\sum_{i=1}^n X_i$ the empirical mean and $S^2 := \frac{1}{n-1}\sum_{i=1}^n (X_i - \overline{M})^2$ the (corrected) sample variance. Check that $\overline{M}$ and S^2 are independent.
[*Hint.* Consider the random vector $(\overline{M}, X_1 - \overline{M}, X_2 - \overline{M}, \dots, X_n - \overline{M})$.

Problem 5.3 a) Let X be a $\mathbb{Z}^d$-valued random variable with characteristic function φ_X. Check that φ_X has period $2\pi\mathbb{Z}^d$, i.e., $\varphi_X(t + 2\pi m) = \varphi_X(t)$ for $t \in \mathbb{R}^d$, $m \in \mathbb{Z}^d$ and

$$\mathbb{P}(X = x) = \frac{1}{(2\pi)^d} \int_{(-\pi,\pi]^d} e^{-i\langle x,t\rangle} \varphi_X(t)\, dt \quad \text{holds for } x \in \mathbb{Z}^d.$$

b) Let X be a real-valued random variable and assume that its characteristic function $\varphi_X \in \mathcal{L}^1(\lambda)$ (with λ the Lebesgue measure on $\mathbb{R}$). Then X has a continuous, bounded density $f_X = \frac{d\mathscr{L}(X)}{d\lambda}$ which is given by

$$f_X(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itx} \varphi_X(t)\, dt.$$

[*Hint.* Consider first $X \sim \mathcal{N}(0, \varepsilon)$, $\varepsilon > 0$, for the general case let $\mu_\varepsilon := \mathscr{L}(X) * \mathcal{N}(0, \varepsilon)$. Check that μ_ε has density f_ϵ and $f_\epsilon \to f_X$ pointwise as $\epsilon \downarrow 0$.]

Problem 5.4 Let $X_1, X_2, \dots$ i.i.d. $\mathbb{Z}$-valued with $\mu := \mathbb{E}[X_1]$ and $0 < \sigma^2 := \mathrm{Var}[X_1] < \infty$, satisfying the aperiodicity condition

$$\gcd\{i - j : i, j \in \mathbb{Z}, \mathbb{P}(X_1 = i)\mathbb{P}(X_1 = j) > 0\} = 1.$$

a) Check that for all $\varepsilon \in (0, \pi)$,

$$\sup_{\varepsilon \leqslant |t| \leqslant \pi} |\varphi_{X_1}(t)| < 1.$$

[*Hint.* Check that $|\varphi_{X_1}(t)| = 1$ implies $t \in 2\pi\mathbb{Z}$.]
b) (A local CLT) Let $S_n := X_1 + \dots + X_n$. Check that for every $K \in (0, \infty)$

$$\lim_{n\to\infty} \sup_{k \in \mathbb{Z}, |k - n\mu| \leqslant K\sigma\sqrt{n}} \Big| \sqrt{2\pi n\sigma^2}\, \mathbb{P}(S_n = k) - \exp\big(-(k - n\mu)^2/(2n\sigma^2)\big) \Big| = 0.$$

[*Hint.* Use discrete Fourier inversion (cf Problem 5.3 a)) and Taylor expansion of $\varphi_{X_1}(t)$ around $t = 0$, using part a) to bound the contributions from t outside a small neighbourhood of 0.]

Please turn over

Problem 5.5 a) The negative binomial distribution with parameters $r > 0$, $p \in (0,1)$ has weights

$$b^{-}_{r,p}(k) = \binom{-r}{k}(-1)^k p^r (1-p)^k, \quad k \in \{0,1,2,\dots\}$$

(with $\binom{-r}{k} = (-r)(-r-1)\cdots(-r-k+1)/k!$; for $r \in \mathbb{N}$ is the number of failures before the r-th success in a p-coin tossing sequences $b^{-}_{r,p}$-distributed). Express $b^{-}_{r,p}$ as CPoi_ν for a suitable $\nu \in \mathcal{M}_f(\mathbb{Z}_+)$ and deduce that $b^{-}_{r,p}$ is infinitely divisible.

[*Hint.* Compute the characteristic function.]

b) Find the canonical triplet of the Gamma distribution $\Gamma_{r,\lambda}$ (whose density is $\frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x} \mathbf{1}_{(0,\infty)}(x)$).

[*Hint.* It suffices to consider $r = 1 = \lambda$ (why?).]

Problem 5.6 a) Let X be a real-valued, infinitely divisible random variable with $\mathbb{P}(|X| \leqslant K) = 1$ for some $K \in (0,\infty)$. Check that X is a.s. constant.

[*Hint.* W.l.o.g. assume $\mathbb{E}[X] = 0$. By assumption, for any $n \in \mathbb{N}$ there are i.i.d. $X_{n,1}, X_{n,2}, \dots, X_{n,n}$ such that $X =^d X_{n,1} + \cdots + X_{n,n}$, hence $\mathbb{P}(X_{n,1} < -K/n) + \mathbb{P}(X_{n,1} > K/n) = 0$; use this to bound the variance of X.]

b) Check that for $\alpha > 2$ there is no probability measure on $\mathbb{R}$ with characteristic function $t \mapsto e^{-|t|^\alpha}$.

[*Hint.* If there was a random X with $\varphi_X(t) = e^{-|t|^\alpha}$, what would its mean and variance be?]

Problem 5.7 Let $\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy$ be the distribution function of the standard normal distribution and define F for $x \in \mathbb{R}$ via

$$F(x) := \begin{cases} 2\big(1 - \Phi(1/\sqrt{x})\big)\,, & \text{if } x > 0, \\ 0\,, & \text{else.} \end{cases}$$

Check that F is the distribution function of a strictly stable distribution $\mu \in \mathcal{M}_1(\mathbb{R}_+)$ of index $1/2$, in particular, μ is infinitely divisible.

[*Hint.* Compute the density of μ and check that its Laplace transform has the form $\lambda \mapsto e^{-\sqrt{2\lambda}}$.]

Abgabe der Aufgaben: 13.1.2015 (Es findet keine Bewertung statt.)