Problem 6.1 Let $\tau_1, \tau_2, \ldots$ be i.i.d., $\sim \mathrm{Exp}(\rho)$, put $T_0 := 0$, $T_n := \tau_1 + \cdots + \tau_n$, $n \in \mathbb{N}$ and let

$$N_t := \sum_{n=1}^{\infty} \mathbf{1}(T_n \leqslant t), \quad t \geqslant 0 \tag{1}$$

($(N_t)_{t\geqslant 0}$ is a Poisson point process with rate ρ). Check that

$$\mathscr{L}(N_{t+h} - N_t) = \mathrm{Poi}_{\rho h} \quad \text{for } t, h \geqslant 0 \tag{2}$$

and that for any $s \geqslant 0$,

$$\tilde{N}_t := N_{t+s} - N_s,\ t \geqslant 0 \quad \text{is a Poisson point process with rate } \rho \text{ and is indep. from } (N_r : r \leqslant s). \tag{3}$$

[*Hint.* The crucial property is the *memorylessness* of the exponential distribution: Check that

$$\mathbb{P}(\tau_i > t + s \mid \tau_i > s) = \mathbb{P}(\tau_i > t) = \int_t^{\infty} \rho e^{-\rho x}\, dx = e^{-\rho t}. \tag{4}$$

Now let $s \geqslant 0$, $m \in \mathbb{N}_0$, consider the event

$$A(s,m) := \{N_s = m\} = \{T_m \leqslant s < T_{m+1}\} = \{T_m \leqslant s\} \cap \{\tau_{m+1} > s - T_m\}.$$

Argue that on $A(s,m)$, $N_r = \sum_{k=1}^{m} \mathbf{1}(T_k \leqslant r)$ for $0 \leqslant r \leqslant s$ and the jump times of $(\tilde{N}_t)$ are $\tilde{T}_1 = \tau_{m+1} - (s - T_m)$, $\tilde{T}_n = T_{n+m} - s$, $n \geqslant 2$. Use this and (4) to verify that for every $m \in \mathbb{N}_0$, conditioned on $A(s,m)$, the sequence $\tilde{T}_1, \tilde{T}_2 - \tilde{T}_1, \tilde{T}_3 - \tilde{T}_2, \ldots$ is i.i.d., $\sim \mathrm{Exp}(\rho)$. This proves claim (3) (why?).

To check (2) note that it suffices to consider $t = 0$ (why?), use the fact that $T_m \sim \Gamma_{m,\rho}$ and $\{N_h = m\} = \{T_m \leqslant h, \tau_{m+1} > h - T_m\}$.

Problem 6.2 a) Let E be a finite set, $Q = (Q_{x,y})_{x,y \in E}$ a generator matrix, i.e., $Q_{x,y} \geqslant 0$ for all $x \neq y$ and $\sum_y Q_{x,y} = 0$ for all x. Then $P(t) := \exp(tQ) = \sum_{k=0}^{\infty} \frac{t^k}{k!} Q^k$, $t \geqslant 0$ defines a semigroup of stochastic matrices. Check that $P(t)$ solves

$$\frac{\partial}{\partial t} P(t) = P(t)Q, \text{ i.e., } \forall\, x, y \in E \ :\ \frac{\partial}{\partial t} P_{x,y}(t) = \sum_z P_{x,z}(t) Q_{z,y} = P_{x,y}(t) Q_{y,y} + \sum_{z \neq y} P_{x,z}(t) Q_{z,y} \tag{5}$$

$$\frac{\partial}{\partial t} P(t) = QP(t), \text{ i.e., } \forall\, x, y \in E \ :\ \frac{\partial}{\partial t} P_{x,y}(t) = \sum_z Q_{x,z} P_{z,y}(t) = \sum_z Q_{x,z} \big(P_{z,y}(t) - P_{x,y}(t)\big), \tag{6}$$

with initial condition $P(0) = I_E$, where I_E is the identity matrix on E.
[*Remark.* (5) are known as Kolmogorov forward equations, (6) as Kolmogorov backward equations.]

b) Let $E = \{0, 1\}$, $(X_t)_{t\geqslant 0}$ a Markov chain on E with generator matrix $Q = \begin{pmatrix} -a & a \\ b & -b \end{pmatrix}$, where $a, b \in (0, \infty)$. Check that

$$\mathbb{P}_0(X_t = 0) = e^{-(a+b)t} + \big(1 - e^{-(a+b)t}\big) \frac{b}{a+b}, \tag{7}$$

$$\mathbb{P}_1(X_t = 1) = e^{-(a+b)t} + \big(1 - e^{-(a+b)t}\big) \frac{a}{a+b}. \tag{8}$$

[*Hint.* Solve for example the Kolmogorov backward equations.]

Please turn over

Problem 6.3 (Discrete martingale problems) Let E be (at most) countable, $p = (p_{x,y})_{x,y \in E}$ a stochastic matrix. For bounded $f : E \to \mathbb{R}$ define $Pf : E \to \mathbb{R}$ via $Pf(x) := \sum_{y \in E} p_{x,y} f(y)$.
a) If $(X_n)_{n \in \mathbb{N}_0}$ is a Markov chain with transition matrix p then for every bounded $f : E \to \mathbb{R}$, the process

$$M_0 := 0, \quad M_n := f(X_n) - f(X_0) - \sum_{k=0}^{n-1} \big(Pf - f\big)(X_k), \quad n \in \mathbb{N} \tag{9}$$

is a martingale (w.r.t. the filtration $\mathcal{F}_n = \sigma(X_0, X_1, \dots, X_n)$) under each $\mathbb{P}_{x_0}$, $x_0 \in E$.

b) The following converse holds: Let $X = (X_n)_{n \in \mathbb{N}_0}$ be a stochastic process with values in E and a family of distributions $\mathbb{P}_{x_0}$, $x_0 \in E$ with $\mathbb{P}_{x_0}(X_0 = x_0) = 1$ so that for every bounded $f : E \to \mathbb{R}$, (9) defines a martingale under each $\mathbb{P}_{x_0}$. Check that then X is a Markov chain with transition matrix p.

[*Hint.* Use (9) with functions $f = 1_{\{y\}}$.]

Problem 6.4 Let $X_1, X_2, \dots$ be i.i.d. $\mathbb{Z}$-valued, $p(x) := \mathbb{P}(X_1 = x) < 1$ for all $x \in \mathbb{Z}$. Put $S_0 := 0$, $S_n := X_1 + \dots + X_n$, $\mathcal{F}_n := \sigma(S_1, \dots, S_n)$, $M_n := \max_{0 \leqslant j \leqslant n} S_j$, $Z_n := M_n - S_n$, $n = 0, 1, 2, \dots$. Check that:

a) $(M_n)_{n=0,1,2,\dots}$ is not a Markov chain (with respect to any filtration).

b) $(M_n, S_n)_{n=0,1,2,\dots}$ is a Markov chain with values in $\mathbb{Z}^2$ (w.r.t. $(\mathcal{F}_n)_{n=0,1,2,\dots}$). Compute its transition probabilities $\mathbb{P}\big((M_{n+1}, S_{n+1}) = (x', y') \,\big|\, (M_n, S_n) = (x, y)\big)$, $x, y, x', y' \in \mathbb{Z}$.

c) $(Z_n)_{n=0,1,2,\dots}$ is a Markov chain with values in $\mathbb{Z}_+$ w.r.t. to $(\sigma(Z_k : k \leqslant n))_{n=0,1,2,\dots}$.

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