Branching processes in random environment – a view on critical and subcritical cases

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Summary. Branching processes exhibit a particularly rich longtime behaviour when evolving in a random environment. Then the transition from subcriticality to supercriticality proceeds in several steps, and there occurs a second 'transition' in the subcritical phase (besides the phase-transition from (sub)criticality to supercriticality). Here we present and discuss limit laws for branching processes in critical and subcritical i.i.d. environment. The results rely on a stimulating interplay between branching process theory and random walk theory. We also consider a spatial version of branching processes in random environment for which we derive extinction and ultimate survival criteria.

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1 Introduction

Branching processes in random environment is one of the topics, which we have studied in the project "Verzweigende Populationen: Genealogische Bäume und räumliches Langzeitverhalten" (Grant Ke 376/6) within the DFG-Schwerpunkt "Interagierende stochastische Systeme von hoher Komplexität". It is representative in that a central aspect of the whole project were probabilistic constructions of genealogical trees and the interplay between branching processes and random walks. These concepts turn out to be significant for branching processes in random environment in a rather specific way. This is due to the fact that from a methodical point of view the subject of branching processes in random environment is substantially influenced by the theory of random walks. Many of the results rely on a stimulating interplay between branching process theory on the one hand and fluctuation theory of random walks on the other hand. Here we present limit theorems for critical and subcritical branching processes in i.i.d. random environment and give a detailed explanation of this relationship.

For classical Galton-Watson branching processes it is assumed that individuals reproduce independently of each other according to some given offspring distribution. In the setting of this paper the offspring distribution varies in a random fashion, independently from one generation to the other. A mathematical formulation of the model is as follows. Let Δ be the space of probability measures on $\mathbb{N}_0$ which equipped with the metric of total variation becomes a Polish space. Let Q be a random variable with values in Δ. Then an infinite sequence $\Pi = (Q_1, Q_2, \ldots)$ of i.i.d. copies of Q is said to form a *random environment*. A sequence of $\mathbb{N}_0$-valued random variables $Z_0, Z_1, \ldots$ is called a *branching process in the random environment* Π, if Z_0 is independent of Π and if given Π the process $Z = (Z_0, Z_1, \ldots)$ is a Markov chain with

$$\mathcal{L}\big(Z_n \mid Z_{n-1} = z, \Pi = (q_1, q_2, \ldots)\big) \;=\; \mathcal{L}\big(\xi_1 + \cdots + \xi_z\big) \tag{1}$$

for every $n \geq 1, z \in \mathbb{N}_0$ and $q_1, q_2, \ldots \in \Delta$, where $\xi_1, \xi_2, \ldots$ are i.i.d. random variables with distribution q_n. In the language of branching processes Z_n is the nth generation size of the population and Q_n is the distribution of the number of children of an individual at generation $n-1$. For convenience, we will assume throughout that $Z_0 = 1$ a.s.

As it turns out the asymptotic properties of Z are first of all determined by its *associated random walk* $S = (S_0, S_1, \ldots)$. This random walk has initial state $S_0 = 0$ and increments $X_n = S_n - S_{n-1}$, $n \geq 1$ defined as

$$X_n \;:=\; \log \sum_{y=0}^{\infty} y\; Q_n(\{y\}) \;,$$

which are i.i.d. copies of the logarithmic mean offspring number

$$X \;:=\; \log \sum_{y=0}^{\infty} y\; Q(\{y\}) \;.$$

We will assume that X is a.s. finite. Due to (1) and our assumption $Z_0 = 1$ a.s. the conditional expectation of Z_n given the environment Π,

$$\mu_n \;:=\; \mathbf{E}[Z_n \mid \Pi] \;,$$

can expressed by means of S as

$$\mu_n \;=\; e^{S_n} \quad \mathbf{P}\text{--}a.s.$$

According to fluctuation theory of random walks (compare Chapter XII in [Fe71]) one may distinguish three types of branching processes in random environment. First, S can be a random walk with positive drift, which means that $\lim_n S_n = \infty$ a.s. In this case $\mu_n \to \infty$ a.s. and Z is called a *supercritical* branching process. Second, S can have negative drift, i.e., $\lim_n S_n = -\infty$ a.s. Then $\mu_n \to 0$ a.s. and Z is called *subcritical*. Finally, S may be an oscillating random walk meaning that $\limsup_n S_n = \infty$ a.s. and at the same time $\liminf_n S_n = -\infty$ a.s., which implies $\limsup_n \mu_n = \infty$ a.s. and $\liminf_n \mu_n = 0$ a.s. Then we call Z a *critical* branching process. Our classification extends the classical distinction of branching processes in random environment introduced in [AK71, SW69]. There it is assumed that the random walk has finite mean. Then Z is supercritical, subcritical or critical according as $\mathbf{E}X > 0$, $\mathbf{E}X < 0$ or $\mathbf{E}X = 0$. Only recently the requirement that the expectation of X exists could be dropped (see [AGKV04, DGV03, VD03]).

The distinction plays a similar role as for ordinary branching processes: If Z is a (non-degenerate) critical or subcritical branching process in random environment, then the population eventually becomes extinct with probability 1. This fact is an immediate consequence of the first moment estimate

$$\mathbf{P}\{Z_n > 0 \mid \Pi\} \;\le\; \mathbf{P}\{Z_m > 0 \mid \Pi\} \;\le\; \mu_m \quad \text{for all } m \le n,$$

which implies

$$\mathbf{P}\{Z_n > 0 \mid \Pi\} \;\le\; \min_{m\le n} \mu_m \;=\; \exp\big(\min_{m\le n} S_m\big). \tag{2}$$

Thus in critical and subcritical cases $\mathbf{P}\{Z_n > 0 \mid \Pi\} \to 0$ a.s. and consequently, $\mathbf{P}\{Z_n \to 0\} = 1$. Other than for classical Galton-Watson branching processes the converse is not always true. Also for supercritical branching processes the random fluctuations of the environment can have the effect that the entire population dies out within only a few generations. A criterion that excludes such catastrophes is the following integrability condition on the conditional probability of having no children (see Theorem 3.1 in [SW69])

$$\mathbf{E}\log\big(1 - Q(\{0\})\big) > -\infty.$$

Finding criteria for ultimate survival becomes a challenging problem for branching processes in random environment with a spatial component. In a branching random walk in space-time i.i.d. random environment individuals have a spatial position (in a countable Abelian group) and move as independent random walkers. Each individual at generation $n-1$ located at x uses offspring law $Q_{n,x}$, where the $Q_{n,x}$ form an i.i.d. random field. The fact that individuals living at different locations use independent offspring distributions leads to a smoothing in comparison to the non-spatial case, and thus one might expect that ultimate survival is easier.

The topic of branching processes in random environment has gone through quite a development. For fairly long time research was restricted to the special case of offspring distributions with a linear fractional generating function (which means that given the event $\{Q \ge 1\}$ the offspring distribution Q is geometric with random mean) and to the case where the associated walk has zero mean, finite variance increments. Under these restrictions fairly explicit (albeit tedious) calculations of certain Laplace transforms are feasible. Later the advantages of methods from the theory of random walks have been recognized. General offspring distributions, however, have become accessible only recently.

In this paper we focus on the longtime behavior of critical and subcritical processes (except for the part on spatial branching processes), where differences to classical branching processes are especially striking. Our discussion is based on a formula for the probability of non-extinction, which is derived in the next section. It is this formula that allows to determine the exact asymptotic magnitude of the probability of non-extinction for general offspring distributions (see [GK00, GKV03, GL01]). The linear fractional case could be treated long before (see [Af80, Ko76]).

In Section 3 we discuss the limiting behaviour of branching processes in critical random environment under a general assumption known as Spitzer's condition in the theory of random walks. Section 4 is devoted to the transition from criticality to subcriticality, which takes place in several steps. If one considers the conditioned branching process given the event $\{Z_n > 0\}$ rather than the unconditioned process, then the transition from (super)criticality to subcriticality occurs within the subcritical phase. This so-called intermediate subcritical case is especially intriguing since it exhibits subcritical as well as supercritical behaviour alternating in time. In Section 5 we derive criteria for the a.s. extinction of a branching random walk in space-time i.i.d. random environment. It is shown that a transient (symmetrized) individual motion can be strong enough to completely counteract the correlations between different individuals introduced by the environment, while a recurrent motion cannot.

Within the DFG-Schwerpunkt "Interagierende stochastische Systeme von hoher Komplexität" related projects are the study of the longtime behaviour of population models in a stationary situation by Greven and by Höpfner and Löcherbach (see their articles in this volume). Also our results display phenomena known from statistical physics (see the models studied in the section "Disordered media" of this volume).

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2 A formula for the survival probability

In this section we investigate the relationship between the conditional survival probabilities $\mathbf{P}\{Z_n > 0 \mid \Pi\}$, $n \geq 0$ and the associated random walk $(S_n)_{n\geq 0}$. An essential observation will be that the estimate (2) not only gives an upper bound but also the right impression of the magnitude of the survival probability,

$$\mathbf{P}\{Z_n > 0 \mid \Pi\} \;\approx\; \min_{m\leq n} \mu_m \;=\; \exp(\min_{m\leq n} S_m). \tag{3}$$

This relation is plausible, however, trying to elaborate it directly to a precise mathematical statement is not a great promise. Instead we reformulate (3). The upper bound (which is estimate (2)) says that $\mathbf{P}\{Z_n > 0 \mid \Pi\}$ becomes particularly small, whenever this is true for some μ_m, $m \leq n$. To put it another way $1/\mathbf{P}\{Z_n > 0 \mid \Pi\}$ gets large, whenever one of the quantities $1/\mu_0, \dots, 1/\mu_n$ is large. It would be particularly useful, if this dependence could be expressed in a linear fashion with coefficients which can be controlled sufficiently well. Thus we are looking for a formula of the form

$$\frac{1}{\mathbf{P}\{Z_n > 0 \mid \Pi\}} \;=\; \sum_{k=0}^{n} \frac{A_{k,n}}{\mu_k}$$

with 'tractable' quantities $A_{k,n} \geq 0$, $0 \leq k \leq n$. In the case of offspring distributions with a linear fractional generating function such a formula had been known for a long time. For general offspring distributions such a representation has been obtained only recently in [GK00].

Here we provide two different approaches to the desired formula. Following [GK00] we first proceed in a purely analytical manner and obtain analytical expressions and estimates for the $A_{k,n}$. Then we will present an alternative derivation of the formula which is obtained by means of a probabilistic construction of the conditional family tree produced by the branching process. This second approach, while leading to the same coefficients $A_{k,n}$, allows a probabilistic interpretation of these terms.

The analytical approach is a straightforward calculation. Consider the (random) generating functions

$$f_k(s) \;:=\; \sum_{y=0}^{\infty} s^y\, Q_k(\{y\}) \;, \quad 0 \leq s \leq 1 \;,$$

$k = 1, 2, \dots$ and their compositions

$$f_{k,n}(s) \;:=\; f_{k+1}(f_{k+2}(\cdots f_n(s)\cdots)) \;, \quad 0 \leq k \leq n,$$

with the convention $f_{n,n}(s) := s$. It follows from (1) and our assumption $Z_0 = 1$ a.s. that $f_{0,n}$ is the conditional generating function of Z_n given the environment Π, i.e.,

$$f_{0,n}(s) \;=\; \mathbf{E}[s^{Z_n} \mid \Pi] \quad a.s. \;, \qquad 0 \le s \le 1 \;. \tag{4}$$

Using a telescope type of argument we deduce

$$\frac{1}{1-f_{0,n}(s)} \;=\; \frac{1}{\mu_n(1-s)} + \sum_{k=0}^{n-1} g_{k+1}(f_{k+1,n}(s)) \frac{1}{\mu_k} \;, \qquad 0 \le s < 1 \;,$$

with

$$g_k(s) \;:=\; \frac{1}{1-f_k(s)} - \frac{1}{f_k'(1)(1-s)} \;, \qquad 0 \le s < 1 \;.$$

By (4), we have $\mathbf{P}\{Z_n > 0 \mid \Pi\} = 1 - f_{0,n}(0)$, so that we end up with the formula

$$\frac{1}{\mathbf{P}\{Z_n > 0 \mid \Pi\}} \;=\; \sum_{k=0}^{n} \frac{A_{k,n}}{\mu_k} \tag{5}$$

with random coefficients

$$A_{k,n} \;:=\; g_{k+1}(f_{k+1,n}(0)) \;,\; 0 \le k < n \qquad \text{and} \qquad A_{n,n} \;:=\; 1 \;.$$

Apparently, this identity has first been utilized by Jirina [Ji76]. By convexity of the f_k, the coefficients $A_{k,n}$ are non-negative. Geiger and Kersting proved the upper bound (see Lemma 2.1 in [GK00])

$$g_k(s) \;\le\; \eta_k \;, \quad 0 \le s < 1 \;, \tag{6}$$

where η_k is the standardized second factorial moment of Q_k,

$$\eta_k \;:=\; \sum_{y=0}^{\infty} y(y-1)Q_k(\{y\}) \Big/ \Big(\sum_{y=0}^{\infty} y Q_k(\{y\}) \Big)^2 \;. \tag{7}$$

Consequently, the $A_{k,n}$, $k < n$ satisfy the estimate

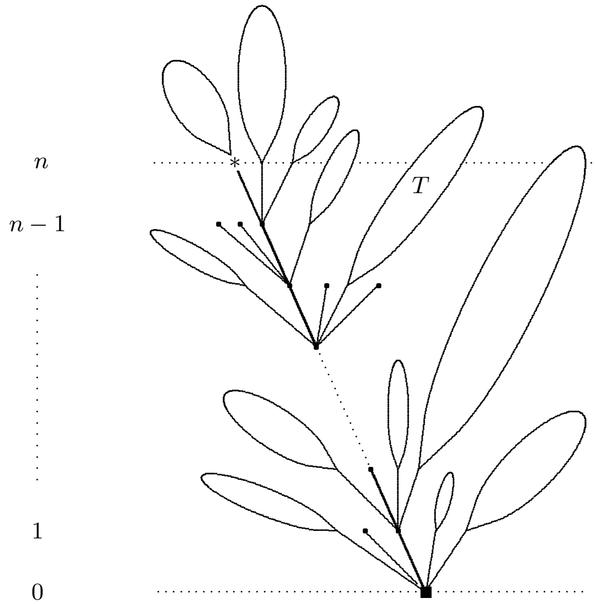
$$0 \;\le\; A_{k,n} \;\le\; \eta_{k+1} \;, \tag{8}$$

which allows to control their magnitude and to exploit the representation of the survival probability in (5).

Thus we succeeded in deriving a formula of the desired form, its probabilistic meaning still has to be revealed. This is achieved by the probabilistic approach which we discuss next. Our starting point is the identity

$$\frac{1}{\mathbf{P}\{Z_n > 0 \mid \Pi\}} \;=\; \frac{\mathbf{E}[Z_n \mid Z_n > 0, \Pi]}{\mathbf{E}[Z_n \mid \Pi]} \;=\; \frac{\mathbf{E}[Z_n \mid Z_n > 0, \Pi]}{\mu_n} \;, \tag{9}$$

which reduces the calculation of the non-extinction probability at n to that of the conditional mean of Z_n. For the latter it is essential to view the branching process as a mechanism to generate a random family tree rather than a mere sequence of generation sizes. We think of the family tree as a rooted ordered



tree with the distinguishable offspring of each individual ordered from left to right. This allows to decompose the family tree along the ancestral line of the left-most individual $*$ at generation n. (In the illustration above the bold line marks the distinguished line of descent starting with the founding ancestor $\blacksquare$.) This picture corresponds to a lucid probabilistic construction of the conditional family tree. The construction was originally devised and investigated for classical Galton-Watson branching processes in [Ge99]. However, the construction works as well for branching processes in varying (deterministic) environment, and, hence, for branching processes in random environment when conditioning on Π and the event $\{Z_n > 0\}$. Note that because of the distinguished role of $*$ as the left-most individual of generation n the subtrees to the left of $*$'s line of descent stay below level n. On the other hand it is natural to expect – and it does follow from the results in [Ge99] – that the subtrees to the right of the distinguished ancestral line remain unaffected by the conditioning event $\{Z_n > 0\}$, i.e., they evolve like ordinary branching processes given Π, independent of other parts of the tree. Thus, if T is one of the subtrees to the right with root in generation $k \in \{1, \dots, n\}$ and $Z_{n,T}$ its number of individuals at generation n, then

$$\mathbf{E}[Z_{n,T} \mid Z_n > 0, \Pi] \;=\; e^{S_n - S_k} \;=\; \frac{\mu_n}{\mu_k}\ .$$

Moreover, if $R_{k,n}$ is the number of siblings to the right of $*$'s ancestor at generation k, then linearity of expectation yields

$$\begin{aligned}\mathbf{E}[Z_n \mid Z_n > 0, \Pi] &= 1 + \mathbf{E}\Big[\sum_T Z_{n,T} \mid Z_n > 0, \Pi\Big] \\ &= 1 + \sum_{k=0}^{n-1} \mathbf{E}[R_{k+1,n} \mid Z_n > 0, \Pi\,] \frac{\mu_n}{\mu_{k+1}} \ ,\end{aligned}$$

where the 1 comes from the distinguished individual $*$ and the first sum extends over all subtrees T to the right of the distinguished ancestral line. Thus, putting

$$\tilde{A}_{k,n} \ := \ \exp(-X_{k+1})\,\mathbf{E}[R_{k+1,n} \mid Z_n > 0, \Pi\,] \ , \ k < n \ , \quad \text{and} \quad \tilde{A}_{n,n} \ := \ 1 \ ,$$

we end up with (recall (9))

$$\frac{1}{\mathbf{P}\{Z_n > 0 \mid \Pi\}} \ = \ \sum_{k=0}^{n} \frac{\tilde{A}_{k,n}}{\mu_k} \ . \qquad (10)$$

It is just a matter of careful calculation to show that representations (5) and (10) agree.

Clearly, $\tilde{A}_{k,n} \geq 0$. An upper estimate for $\tilde{A}_{k,n}$ may be derived as follows. The construction of the conditional family tree in [Ge99] shows that the ancestor of $*$ at generation k belongs to a family of a size, which is stochastically bounded by the so-called size-biased distribution

$$\widetilde{Q}_k(\{y\}) \ := \ \frac{y\, Q_k(\{y\})}{\exp(X_k)} \ , \quad y \geq 0.$$

It follows

$$\begin{aligned}\mathbf{E}[R_{k,n} \mid Z_n > 0, \Pi] &\leq \ \sum_y y\, \widetilde{Q}_k(\{y\}) - 1 \\ &= \ \exp(-X_k) \sum_y y(y-1) Q_k(\{y\}) \ .\end{aligned}$$

Thus

$$0 \ \leq \ \tilde{A}_{k,n} \ \leq \ \exp(-2X_{k+1}) \sum_y y(y-1) Q_{k+1}(\{y\}),$$

which is relation (8). We note that even though the estimate (8) can be verified in a purely analytical manner it took the probabilistic interpretation to find it.

3 Criticality

As explained in the introduction a branching population in a (sub)critical random environment eventually becomes extinct with probability 1. A fundamental question for these processes is the following: If the population survives until some late generation n, in which way does this event occur. One can imagine several ways: The population might have been lucky to find an extraordinarily favourable environment, in which chances for survival are high. Or the population evolved in a typical environment, still by good luck it managed to avoid extinction. In order to weigh these alternatives the formula

$$\mathbf{P}\{Z_n > 0 \mid \Pi\} \;=\; \Big(\sum_{k=0}^{n} A_{k,n} e^{-S_k}\Big)^{-1}$$

from the last section proves useful. For the moment let us neglect effects coming from the $A_{k,n}$, which are of secondary order. Then, as explained above, the probability of survival is high for those unlikely environments, for which $\min(S_0,\dots,S_n)$ is close to 0, otherwise the probability is extremely small.

In this section we look at critical branching processes, i.e., at the case of an oscillating associated random walk. Then the probability of the event $\{\min(S_0,\dots,S_n) \ge 0\}$ is typically of order $n^{-\gamma}$ for some $\gamma > 0$. Thus exponentially small probabilities are negligible and one has to take only those ways of survival into account, where the associated random walk stays away from low negative values (for precise results compare Theorem 1 below and its corollary). These heuristic considerations suggest a program of research, which has been initiated by Kozlov [Ko76] and followed up by several authors, see [Af93, Af97, Af01b, DGV03, GK00, Ko95, Va02].

The results of this section and parts of the discussion are adapted from the recent paper [AGKV04]. As an overall assumption let us adopt **Spitzer's condition** from fluctuation theory of random walks: Assume that there exists a number $0 < \rho < 1$ such that

$$\frac{1}{n}\sum_{m=1}^{n} \mathbf{P}\{S_m > 0\} \;\to\; \rho \; .$$

This general condition guarantees that S is a non-degenerate oscillating random walk. Not striving for greatest generality we restrict ourselves here to two alternative sets of transparent further assumptions. Our first set of assumptions strengthens the condition on the associated random walk and adds some integrability condition on the standardized second factorial moment η (recall the definition in (7)).

Assumption (A). *Let the distribution of X belong without centering to the domain of attraction of a stable law λ with index $\alpha \in (0,2]$. The limit law λ is not a one-sided stable law, i.e., $0 < \lambda(\mathbb{R}^+) < 1$. Further let for some $\epsilon > 0$*

$$\mathbf{E}\,(\log^+ \eta)^{\alpha+\epsilon} \;<\; \infty \; .$$

Note that $\alpha = 2$ is the case of a non-degenerate zero mean, finite variance random walk.

If we assume specific types of distribution for Q, then we can relax Assumption (A). In particular, we can deal with the three most important special cases of offspring distributions (playing a prominent role for classical branching processes, too).

Assumption (B). *The random offspring distribution Q is a.s. a binary, a Poisson or a geometric distribution on $\mathbb{N}_0$ (with random mean).*

The first of our results on the longtime behaviour of branching processes in critical random environment concerns the asymptotic behaviour of the survival probability at n (for the proofs of the results of this section we refer to the paper [AGKV04]).

Theorem 1. *Assume (A) or (B). Then there exists a number $0 < \theta < \infty$ such that, as $n \to \infty$,*

$$\mathbf{P}\{Z_n > 0\} \ \sim \ \theta\, \mathbf{P}\{\min(S_0, \dots, S_n) \geq 0\} \ .$$

This theorem gives evidence for our claim that the behaviour of Z is primarily determined by the random walk S. Only the constant θ depends on the fine structure of the random environment.

Since under Spitzer's condition the asymptotic behaviour of the probability on the right-hand side above is well-known, we obtain the following corollary.

Corollary 1. *Assume (A) or (B). Then, as $n \to \infty$,*

$$\mathbf{P}\{Z_n > 0\} \ \sim \ \theta n^{\rho - 1} l(n) \ ,$$

where $l(1), l(2), \dots$ is a sequence varying slowly at infinity.

The next theorem shows that conditioned on the event $\{Z_n > 0\}$ the process $Z_0, Z_1, \dots, Z_n$ exhibits 'supercritical behaviour'. Supercritical branching processes (whether classical or in random environment) obey the growth law $Z_n/\mu_n \to W$ a.s., where W is some typically non-degenerate random variable. In our situation this kind of behaviour can no longer be formulated as a statement on a.s. convergence, since the conditional probability measures depend on n.

Instead, we define for integers $0 \leq r \leq n$ the process $X^{r,n} = (X_t^{r,n})_{0 \leq t \leq 1}$ given by

$$X_t^{r,n} \ := \ \frac{Z_{r+[(n-r)t]}}{\mu_{r+[(n-r)t]}} \ , \quad 0 \leq t \leq 1 \ . \tag{11}$$

Theorem 2. *Assume (A) or (B). Let $r_1, r_2, \dots$ be a sequence of natural numbers such that $r_n \leq n$ and $r_n \to \infty$. Then, as $n \to \infty$,*

$$\mathcal{L}\big(X^{r_n,n} \,\big|\, Z_n > 0\big) \ \Longrightarrow \ \mathcal{L}\big((W_t)_{0 \leq t \leq 1}\big) \ ,$$

where the limiting process is a stochastic process with a.s. constant paths, i.e., $\mathbf{P}\{W_t = W \text{ for all } t \in [0,1]\} = 1$ for some random variable W. Furthermore,

$$\mathbf{P}\{0 < W < \infty\} \ = \ 1 \ .$$

The symbol $\Longrightarrow$ denotes weak convergence w.r.t. the Skorokhod topology in the space $D[0,1]$ of càdlàg functions on the unit interval. Again the growth of Z is in the first place determined by the random walk (namely, the sequence $(\mu_k)_{0\le k\le n}$). The fine structure of the random environment is reflected only in the distribution of W.

Thus the properties of S determine the behaviour of Z in the main. On the other hand one has to take into account that the properties of the random walk change drastically, when conditioned on the event $\{Z_n > 0\}$. As explained above one expects that S conditioned on $\{Z_n > 0\}$ behaves just as S given the event $\{\min(S_0, \dots, S_n) \ge 0\}$, i.e., like a random walk conditioned to stay positive for a certain period of time (a so-called random walk meander). The next theorem confirms this expectation. Here we need Assumption (A).

Theorem 3. *Assume (A). Then there exists a slowly varying sequence* $\ell(1), \ell(2), \dots$ *such that, as* $n \to \infty$*,*

$$\mathcal{L}\Big(\big(n^{-\frac{1}{\alpha}}\ell(n)S_{[nt]}\big)_{0\le t\le 1} \,\Big|\, Z_n > 0\Big) \quad \Longrightarrow \quad \mathcal{L}(L^+) \ ,$$

where L^+ *denotes the meander of a stable Lévy process with index* α*.*

Shortly speaking the meander $L^+ = (L_t^+)_{0\le t\le 1}$ is a stable Lévy process L conditioned to stay positive for $0 < t \le 1$ (for details we refer to [AGKV04]). In view of Theorem 2 the assertion of Theorem 3 is equivalent to the following result.

Corollary 2. *Assume (A). Then, as* $n \to \infty$*,*

$$\mathcal{L}\Big(\big(n^{-\frac{1}{\alpha}}\ell(n)\,\log Z_{[nt]}\big)_{0\le t\le 1} \,\Big|\, Z_n > 0\Big) \quad \Longrightarrow \quad \mathcal{L}(L^+)$$

for some slowly varying sequence $\ell(1), \ell(2), \dots$

4 A transition within the subcritical phase

We have seen in Theorem 2 that critical branching processes in random environment exhibit supercritical behaviour, when conditioned on non-extinction. This phenomenon does not vanish instantly in the subcritical phase, but persists for processes in the 'vicinity' of criticality. The transition from criticality to subcriticality proceeds in several steps in a fashion, which is not known for ordinary branching processes. From now on we assume that the conditional mean offspring number has finite moments of all orders t,

$$\mathbf{E}\exp(tX) \ < \ \infty \quad , \quad t \ge 0 \ .$$

In particular, X^+ has finite mean and subcriticality corresponds to

$$\mathbf{E}X \ < \ 0 \ .$$

In this case the probability of non-extinction at generation n no longer decays at a polynomial rate (as in the critical case) but at an exponential rate. Dealing with exponentially small probabilities it is natural to consider the 'tilted' measures $\widehat{\mathbf{P}}_\beta$, $\beta \geq 0$ given by

$$\widehat{\mathbf{E}}_\beta[\phi(Q_1, \dots, Q_n, Z_1, \dots, Z_n)] \ := \ \gamma^{-n}\mathbf{E}[\phi(Q_1, \dots, Q_n, Z_1, \dots, Z_n)e^{\beta S_n}]$$

for non-negative test functions ϕ, where

$$\gamma \ = \ \gamma_\beta \ := \ \mathbf{E}[e^{\beta X}] \ .$$

For the probability of survival at n we obtain

$$\mathbf{P}\{Z_n > 0\} \ = \ \gamma^n \widehat{\mathbf{E}}_\beta[\mathbf{P}\{Z_n > 0 \mid \Pi\}e^{-\beta S_n}] \ , \quad n \geq 0 \ .$$

For a suitable choice of β let us proceed in a heuristic fashion and replace $\mathbf{P}\{Z_n > 0 \mid \Pi\}$ by the upper bound $\exp(\min(S_0, \dots, S_n))$, which, as we have argued in Section 2, is typically of the same order. Thus we consider

$$\mathbf{E}[e^{\min(S_0,\dots,S_n)}] \ = \ \gamma^n \widehat{\mathbf{E}}_\beta[e^{\min(S_0,\dots,S_n)-\beta S_n}] \ . \tag{12}$$

Following common strategies from large deviation theory we like to choose β in such a way that the expectation on the right-hand side of (12) no longer decays (nor grows) at an exponential rate. To keep the quantity $\min(S_0, \dots, S_n) - \beta S_n$ bounded from above, we only consider the range $\beta \leq 1$. There are three different scenarios where $\min(S_0, \dots, S_n) - \beta S_n$ is close to 0 with sufficiently large probability:

(S1) $\widehat{\mathbf{E}}_\beta X = 0$ with $\beta < 1$. Then $\min(S_0, \dots, S_n) - \beta S_n$ takes a value close to 0 if and only if both $\min(S_0, \dots, S_n)$ and S_n are close to 0. It is known that for a zero mean, finite variance random walk the probability of the event $\{S_0, \dots, S_{n-1} \geq 0, S_n \leq 0\}$ (the probability for a random walk excursion of length n) is of order $n^{-3/2}$. Thus in this case one expects

$$\widehat{\mathbf{E}}_\beta[e^{\min(S_0,\dots,S_n)-\beta S_n}] \ \approx \ n^{-3/2} \ .$$

(S2) $\widehat{\mathbf{E}}_\beta X = 0$ with $\beta = 1$. Then $\min(S_0, \dots, S_n) - S_n$ is close to 0, if $\min(S_0, \dots, S_n)$ and S_n are close to each other, which essentially means that the path $S_0, \dots, S_n$ attains its minimum close to the end. For a zero mean, finite variance random walk the probability of the event $\{S_0, \dots, S_{n-1} \geq S_n\}$ is of order $n^{-1/2}$. Thus in this case one expects

$$\widehat{\mathbf{E}}_1[e^{\min(S_0,\dots,S_n)-S_n}] \ \approx \ n^{-1/2} \ .$$

(S3) $\widehat{\mathbf{E}}_1 X < 0$. Then it is to be expected that

$$\widehat{\mathbf{E}}_1[e^{\min(S_0,\dots,S_n)-S_n}] \ \approx \ \text{const} \ ,$$

since for a random walk with negative drift the quantity $\min(S_0, \dots, S_n) - S_n$ is of constant order.

Each of these scenarios might occur for branching processes in random environment. Our discussion above shows that the we have to distinguish three cases depending on where the parameter $\widehat{\mathbf{E}}_1 X = \gamma^{-1}\mathbf{E}[Xe^X]$ is located with respect to the value 0 (note that $\widehat{\mathbf{E}}_\beta X$ increases with β). In the sequel we will discuss each case in detail. Some of the results to follow are contained in manuscripts submitted for publication, some are proved in special situations and other are part of research in progress. We seize the opportunity and give a general prospect of the results rather than to go into technical details or to state precise integrability conditions (for single results see [Af80, Af98, Af01a, Af01b, De88, d'SH97, FV99, GKV03, GL01, Liu96]).

4.1 The weakly subcritical case.

First we assume

$$\mathbf{E}X < 0 , \quad \mathbf{E}[Xe^X] > 0 . \tag{13}$$

Then there exists a number $0 < \beta < 1$ such that

$$\mathbf{E}[Xe^{\beta X}] = 0 ,$$

which is our choice for the parameter of the tilted measure. Thus

$$\widehat{\mathbf{E}}_\beta X = 0 ,$$

and we are in the scenario described under (S1). In accordance with our discussion there the survival probability obeys the following asymptotics.

Result 4.1. *Assume (13). Then there exists a number* $0 < \theta < \infty$ *such that, as* $n \to \infty$,

$$\mathbf{P}\{Z_n > 0\} \sim \theta\, n^{-3/2}\gamma^n .$$

Moreover, our heuristic arguments from (S1) suggest that only those environments give an essential contribution to the probability of non-extinction whose associated random walk is (close to) an excursion. The following result confirms this expectation.

Result 4.2. *Assume (13) and let* $\sigma^2 := \widehat{\mathbf{E}}_\beta X^2$*. Then, as* $n \to \infty$,

$$\mathcal{L}\Big(\big(\sigma n^{-\frac{1}{2}} S_{[nt]}\big)_{0\le t\le 1} \,\Big|\, Z_n > 0\Big) \implies \mathcal{L}(B^e) ,$$

where B^e *denotes a standard Brownian excursion of length 1, i.e., a Brownian motion* $B = (B_t)_{0\le t\le 1}$ *given the event* $\{B_t \ge 0 \text{ for all } 0 \le t < 1, B_1 = 0\}$.

This result is in some sense similar to Theorem 3 so that again one might expect that the branching process exhibits supercritical behaviour, as long as the random walk excursion is still far away from 0. The following result, which is the analogue of Theorem 2, says that this is indeed true. In fact, this behaviour persists until just before the random walk's return to 0. Recall the definition of the process $X^{r,n}$ from (11).

Result 4.3. *Assume (13). Let $r_1, r_2, \dots$ be a sequence of natural numbers such that $r_n \le n/2$ and $r_n \to \infty$. Then, as $n \to \infty$,*

$$\mathcal{L}\big(X^{r_n, n-r_n} \,\big|\, Z_n > 0\big) \implies \mathcal{L}\big((W_t)_{0 \le t \le 1}\big),$$

where the limiting process is a stochastic process with a.s. constant paths, i.e., $\mathbf{P}\{W_t = W$ for all $t \in [0,1]\} = 1$ for some random variable W. Furthermore,

$$\mathbf{P}\{0 < W < \infty\} \;=\; 1 \;.$$

4.2 The intermediate subcritical case.

This case is where supercritical behaviour resolves into subcritical behaviour on the event $\{Z_n > 0\}$. Here we assume

$$\mathbf{E}X \;<\; 0 \;, \quad \mathbf{E}[Xe^X] \;=\; 0 \;. \tag{14}$$

This time we choose $\beta = 1$, so that

$$\widehat{\mathbf{E}}_\beta X \;=\; 0 \;.$$

Now (S2) is the relevant scenario and the behaviour of the process changes in a remarkable fashion.

Our first result again describes the exact asymptotics of the non-extinction probability at n.

Result 4.4. *Assume (14). Then there exists a number $0 < \theta < \infty$ such that, as $n \to \infty$,*

$$\mathbf{P}\{Z_n > 0\} \;\sim\; \theta\, n^{-1/2} \gamma^n \;.$$

As suggested in the discussion of (S2) the event of survival is essentially carried by those environments whose associated random walk path has a late minimum. This is expressed by the following limit law.

Result 4.5. *Assume (14). Then, as $n \to \infty$,*

$$\mathcal{L}\Big(\big(\sigma n^{-\frac{1}{2}} S_{[nt]}\big)_{0 \le t \le 1} \,\Big|\, Z_n > 0\Big) \implies \mathcal{L}(B^l) \;,$$

where B^l denotes a standard Brownian motion $B = (B_t)_{0 \le t \le 1}$ conditioned on the event $\{B_t \ge B_1$ for all $0 \le t \le 1\}$.

The change from Brownian excursion B^e to conditional Brownian motion B^l implies a change in the behaviour of Z which is unique for branching processes. Note that B^l is not built of a single excursion but contains countably many local excursions. At the same time B^l has (uncountably) many local minima. For this reason we distinguish two different types of epochs. The moment $k \in \{0, 1, \dots, n\}$ is of the first type if the random walk S reaches a new minimum around k. Such moments are particularly difficult to survive and an unconditioned population would die out in the long run when repeatedly facing such bottlenecks. Given the event $\{Z_n > 0\}$ it is natural to expect that at such epochs the population consists of only very few indviduals (as in the strongly subcritical case discussed below). The other type of epoch is if the last minimum of S before time k is quite some time in the past and the next minimum is still quite far in the future. Then the random walk is in the midth of a local excursion. On such time stretches the population again exhibits supercritical behaviour and grows to a large size (as in the preceding weakly subcritical case). Thus it is reasonable to expect that sub- and supercritical behaviour alternate in the course of time.

It requires some effort to convert these heuristics into a precise mathematical statement. Still we formulate a result corresponding to Theorem 2 and Result 4.3. Keeping track of the successive minima of S the right normalization of Z_k is seen to be

$$\tilde{\mu}_k \ := \ \frac{\mu_k}{\min_{j \le k} \mu_j} \ = \ \exp\big(S_k - \min_{j \le k} S_j\big)$$

(rather than μ_k). (3) shows that one might alternatively use

$$\bar{\mu}_k \ := \ \frac{\mu_k}{\mathbf{P}\{Z_k > 0 \mid \Pi\}} \ = \ \mathbf{E}[Z_k \mid Z_k > 0, \Pi] \ .$$

Now recall that an excursion interval of the conditioned Brownian process B^l is an interval $e = (a, b) \subset [0, 1]$ of maximal length such that $B^l_a = B^l_b$ and $B^l_t > B^l_a$ for all $t \in e$. There are countably many excursion intervals $e_1, e_2, \dots$ which we assume to be enumerated in some order. Write $j(t) := j$, if $t \in e_j$.

Result 4.6. *Assume (14) and let* $0 < t_1 < \cdots < t_k < 1$. *Then, as* $n \to \infty$,

$$\mathcal{L}\Big(\Big(\frac{Z_{[nt_1]}}{\tilde{\mu}_{[nt_1]}}, \dots, \frac{Z_{[nt_k]}}{\tilde{\mu}_{[nt_k]}}\Big) \Bigm| Z_n > 0\Big) \ \xrightarrow{w} \ \mathcal{L}\big((W_{j(t_1)}, \dots, W_{j(t_k)})\big) \ ,$$

where $W_1, W_2, \dots$ *are independent of* B^l *and i.i.d. copies of some random variable* W *satisfying*

$$\mathbf{P}\{0 < W < \infty\} \ = \ 1 \ .$$

Thus the ith and jth component of the limiting random vector are identical, if t_i and t_j belong to the same excursion interval of B^l, otherwise they are independent. This result expresses the alternation between sub- and supercritical behaviour described above.

4.3 The strongly subcritical case.

Finally, we come to the case, where supercritical behaviour vanishes completely. Now let

$$\mathbf{E}X \ < \ 0 \ , \quad \mathbf{E}[Xe^X] \ < \ 0 \ . \tag{15}$$

We choose $\beta = 1$ again. Then (other than in the intermediate subcritical case)

$$\widehat{\mathbf{E}}_\beta X \ < \ 0$$

and we are in the situation captured in scenario (S3).

Result 4.7. *Assume (15). Then there exists a number $0 < \theta \leq 1$ such that, as $n \to \infty$,*

$$\mathbf{P}\{Z_n > 0\} \ \sim \ \theta\gamma^n \ .$$

In scenario (S3) the behaviour of S given non-extinction at n is governed by the law of large numbers. This is the content of the following result.

Result 4.8. *Assume (15). Then, as $n \to \infty$,*

$$\mathcal{L}\Big(\big(n^{-1}S_{[nt]}\big)_{0\leq t\leq 1} \,\Big|\, Z_n > 0\Big) \ \Longrightarrow \ \mathcal{L}\big((t\,\widehat{\mathbf{E}}_1 X)_{0\leq t\leq 1}\big) \ .$$

In particular, local random walk excursions vanish in the scaling limit and Z no longer exhibits any supercritical behaviour on the event $\{Z_n > 0\}$. Instead our last result shows that the population stays small throughout the time interval from 0 to n.

Result 4.9. *Assume (15) and let $0 < t_1 < \cdots < t_k < 1$. Then, as $n \to \infty$,*

$$\mathcal{L}\big((Z_{[nt_1]}, \ldots, Z_{[nt_k]}) \,\big|\, Z_n > 0\big) \ \xrightarrow{d} \ (W_1, \ldots, W_k) \ ,$$

where $W_1, W_2, \ldots$ are i.i.d. copies of some random variable W satisfying

$$\mathbf{P}\{1 \leq W < \infty\} \ = \ 1 \ .$$

5 Spatial branching processes in space-time i.i.d. random environment

In this section we consider a model of a spatial branching process in random environment. Now individuals live on a countable Abelian group G. Again we start with a single founding ancestor who at time 0 is located at position $0 \in G$. Individuals at generation $n-1$ located at x have independent offspring according to the random distribution $Q_{n,x}$. Each child independently moves to y with probability $p(x,y) = p(y-x)$, where p is a given (irreducible) random walk kernel. Let $Z_n(x)$ be the number of individuals at x in generation n, and $Z_n := \sum_x Z_n(x)$ the population size at generation n.

The random offspring distributions $Q_{n,x}$, $n \in \mathbb{N}$, $x \in G$ are assumed i.i.d. Given the environment $\Pi = (Q_{n,x})$ individuals branch and move independently. For a probability measure $q = (q_y)_{y\in\mathbb{N}_0}$ we denote the first and second moments as

$$m_1(q) := \sum_y y q_y, \quad m_2(q) := \sum_y y^2 q_y.$$

We will assume $\mathbf{E} m_2(Q) < \infty$. A quantity of particular interest is

$$m := \mathbf{E} m_1(Q), \tag{16}$$

the mean number of offspring per individual. For *deterministic* Q, where $(Z_n)_{n\geq 0}$ is a classical Galton-Watson process, the case $m = 1$ is critical in the sense that $Z_n \to 0$ a.s., if $m \leq 1$ (and $q_1 < 1$), whereas $Z_n \to \infty$ with positive probability if $m > 1$. On the other hand, we have seen in previous sections that in a *non-spatial* scenario with $G = \{0\}$, the criterion for criticality is $\mathbf{E} \log m_1(Q) = 0$. We shall see that the model considered here is in a certain sense intermediate between these two cases. In principle, it is a special case of branching processes in random environment with infinitely many types.

Let $\mathcal{F}_n := \sigma(Z_k(x), Q_{k+1,x}, x \in G, k \leq n)$. One easily checks that

$$M_n := \frac{Z_n}{m^n}, \quad n = 0, 1, \ldots \tag{17}$$

is an $(\mathcal{F}_n)$-martingale.

Let X and Y be two independent p-random walks on G. Note that then $X - Y$ is again a random walk, we denote its transition matrix by $\tilde{p}(x) := \sum_y p(y)p(x-y)$. Let

$$\alpha := \frac{\mathbf{E} m_1(Q)^2}{\big(\mathbf{E} m_1(Q)\big)^2} \quad (\in (1,\infty)\,), \tag{18}$$

$$\alpha_* := \sup\Big\{ a > 0 : \mathbf{E}_{(0,0)}\big[a^{\#\{i\geq 1: Y_i = X_i\}}\,|X\big] < \infty \text{ a.s.}\Big\}. \tag{19}$$

Remark. Obviously, $\alpha_* \geq \big(\mathbf{P}_{(0,0)}(\exists\, k \geq 1 : X_k = Y_k)\big)^{-1}$. This inequality is in fact strict in many cases, as

$$\alpha_* = 1 + \Bigg(\sum_{n\geq 1} \exp(-H(p^n))\Bigg)^{-1}$$

whenever $\sup_{n\geq 1, x\in G} p^n(x)/\tilde{p}^n(0) < \infty$, where $H(p^n)$ is the entropy of $p^n(0,\cdot)$, see Theorem 5 in [Bi03].

Theorem 4. *a) If $m \le 1$ we have $Z_n \to 0$ a.s., in particular, $M_\infty = 0$.*
b) Assume that $\mathrm{Var}(m_1(Q)) > 0$ *and that there exists a sequence (C_n) of finite subsets of G satisfying*

$$\sum_n |C_n|^{-1} = \infty \quad \text{and} \quad \lim_{n\to\infty} \sum_{y\in C_n} p^n(y,0) = 1. \tag{20}$$

Then $M_\infty = 0$, irrespective of the value of m.
c) If $m > 1$, $\tilde{p}$ is transient, $\alpha < \alpha_*$, and

$$\liminf_{k\to\infty} m^k p^k(0, X_k) > 0 \quad a.s., \tag{21}$$

then the family (M_n) is uniformly integrable, hence, $\mathbf{E}\, M_\infty = 1$. In particular, $Z_n \to \infty$ with positive probability in this case.

Remark. Condition (20) in b) implies that $\tilde{p}$ is recurrent. It is satisfied for $G = \mathbb{Z}$ if $\sum_x |x| p(x) < \infty$, and for $G = \mathbb{Z}^2$ if $\sum_x ||x||^2 p(x) < \infty$ (see [Li85], p. 450).

Condition (21) is satisfied (in fact, the left-hand side is ∞) whenever p satisfies a local CLT and a LIL, so e.g. for simple random walk on $\mathbb{Z}^d$.

Sketch of proof. a) Follows easily by comparison with a classical Galton-Watson process with offspring generating function $\bar{\varphi}(s) := \mathbf{E}[\sum_y s^y Q(\{y\}]$: A Jensen-type argument gives

$$\mathbf{E}[s^{Z_n}] \ge \underbrace{\bar{\varphi} \circ \cdots \circ \bar{\varphi}}_{n \text{ times}}(s), \quad s \in [0,1],$$

and it is well-known that the right-hand side tends to 1 as $n \to \infty$ because $\bar{\varphi}'(1-) \le 1$. Thus $Z_n \to 0$ in probability, which together with $\{Z_n = 0\} \subset \{Z_m = 0,\ \forall\, m \ge n\}$ proves the claim.

b) Note that $\zeta_n(x) := m^{-n} \mathbf{E}\big[Z_n(x) \,|\, \mathit{\Pi}\big]$ satisfies

$$\zeta_{n+1}(x) = \sum_y \zeta_n(y) \frac{m_1(Q_{n+1,y})}{m} p(y,x), \quad n \in \mathbb{N}_0, x \in G, \tag{22}$$

so (ζ_n) is a discrete-time version of a Potlatch process (cf. e.g. Chapter IX in [Li85]), starting from $\zeta_0(\cdot) = \delta(0,\cdot)$. One can easily adapt the proof of Theorem IX.4.5 in [Li85] to this discrete-time setting to obtain that $\mathbf{E}[M_n |\, \mathit{\Pi}\,] = \sum_x \zeta_n(x) \to 0$ a.s. as $n \to \infty$ if there is a sequence (C_n) satisfying (20). This implies that $M_n \to 0$ in probability by the (conditional) Markov inequality. As $M_n \to M_\infty$ a.s. we obtain $\mathbf{P}(M_\infty = 0) = 1$.

c) It is well-known that uniform integrability of $(M_n)_{n\in\mathbb{N}_0}$ is equivalent to tightness of the family of the corresponding size-biased laws, see e.g. Lemma 9 in [Bi03]. We use a stochastic representation of the size-biasing of M_n to obtain the criterion. We can view $\mathbf{P}$ as a measure on {ordered, spatially embedded trees} $\times$ {space-time fields of offspring distributions}. We think of size-biasing of Z_n as picking "uniformly" an individual at time n from a(n infinite) forest of independent trees (grown in independent environments), and then looking at the tree the chosen individual belongs to. Let us denote the chosen (spatially embedded, ordered) tree by τ with selected ancestral line λ, denote the space-time field of offspring laws by $Q_{n,x}$, $x \in G, n \in \mathbb{N}$. Technically, we construct a measure $\tilde{\mathbf{P}}$ on {infinite, ordered, spatially embedded trees} $\times$ {ancestral lines} $\times$ {space-time fields of offspring distributions} with the property

$$\begin{aligned}&\tilde{\mathbf{P}}\big(\tau|_n = t, \lambda|_n = a, Q_{k,x} \in B_{x,k}, x \in A,\, 1 \le k \le n+1\big)\\ &\quad = \frac{1}{m^n}\mathbf{P}\big(\tau|_n = t, Q_{k,x} \in B_{x,k}, x \in A,\, 1 \le k \le n+1\big)\end{aligned} \tag{23}$$

for any $n \in \mathbb{N}$ and spatially embedded, ordered, rooted tree t of height at most n, ancestral line $a \subset t$ of length n, finite $A \subset G$ and measurable $B_{x,k} \subset \mathcal{M}_1(\mathbb{N}_0)$. $(\tau, \lambda, \mathit{\Pi})$ under $\tilde{\mathbf{P}}$ arises as follows:

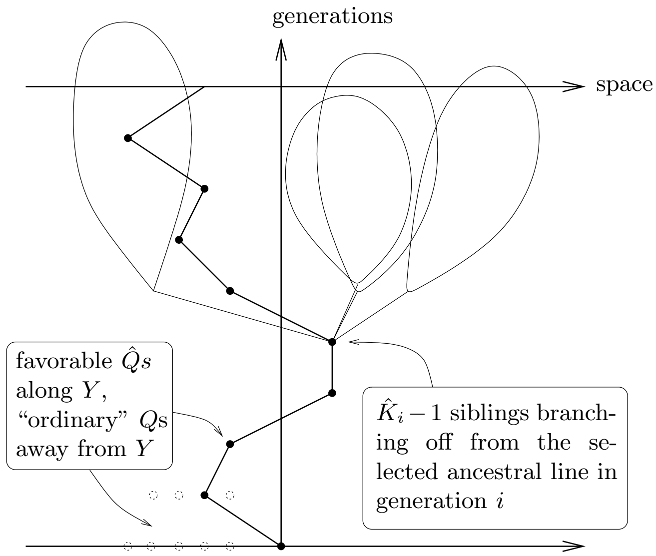
Let $Y = (Y_j)$ be a p-random walk starting from $Y_0 = 0$. This will be the spatial embedding of the selected line. Given Y, the field $\mathit{\Pi}$ has independent coordinates, and the law of $Q_{j+1,x}$ is $\mathcal{L}(Q)$ if $x \neq Y_j$, whereas it is $\mathcal{L}(\hat{Q})$ if $x = Y_j$, where $\tilde{\mathbf{P}}(\hat{Q} \in d\nu) = (m_1(\nu)/m)\mathbf{P}(Q \in d\nu)$. Given this, let $\hat{K}_0, \hat{K}_1, \dots$ be independent with $\tilde{\mathbf{P}}(\hat{K}_i = k|Y, \mathit{\Pi}) = (k/m_1(Q_{i+1,Y_i}))Q_{i+1,Y_i}(\{k\})$. The individual at generation i along the selected line will have $\hat{K}_i$ children (note that $\hat{K}_i \ge 1$ always), and we choose uniformly one of them to continue the distinguished line. The spatial embedding of the chosen child will then be Y_{i+1}, her siblings take each an independent p-step from Y_i. Finally, all the siblings branching off from the selected line form independent populations in the given space-time medium $\mathit{\Pi}$ they see. Then straightforward calculation gives (23). We omit the details, see [Bi03], p. 69f where an analogous proof is carried out for a related construction. By summing over all possible ancestral lines of length n we obtain from (23) that

$$\tilde{\mathbf{P}}(Z_n = k) = \frac{k}{m^n}\mathbf{P}(Z_n = k).$$

We have to show that the distributions of M_n under $\tilde{\mathbf{P}}$ form a tight family. In order to do so it suffices to show that

$$\sup_{n \in \mathbb{N}} \tilde{\mathbf{E}}\big[M_n \,|\, Y\big] < \infty \quad \text{a.s.} \tag{24}$$

To prove (24) we compute



$$\tilde{\mathbf{E}}\left[\frac{Z_n}{m^n}\middle| Y\right] = \frac{1}{m^n}\left\{1 + \mathbf{E}[\hat{K}-1]\sum_{k=0}^{n-1}\mathbf{E}_{(Y_k,k)}\left[\left(\mathbf{E}[m_1(\hat{Q})\right)^{\#\{k<i<n:Y_i=X_i\}} \times m^{\#\{k<i<n:Y_i\neq X_i\}}\middle| Y\right]\right\}$$

$$= \frac{1}{m^n} + \sum_{k=0}^{n-1}\frac{\mathbf{E}[\hat{K}-1]}{m^{k+1}}\mathbf{E}_{(Y_k,k)}\left[\alpha^{\#\{k<i<n:Y_i=X_i\}}\middle| Y\right].$$

Note that assumption (21) implies that

$$A := \inf_{k\in\mathbb{N}} m^{k+1}p_k(0,Y_k) \in (0,\infty] \quad \text{a.s.,} \tag{25}$$

which allows to estimate

$$\frac{1}{m^k}\mathbf{E}_{(Y_k,k)}\left[\alpha^{\#\{k<i<n:Y_i=X_i\}}\middle| Y\right] \le \frac{1}{A}\mathbf{E}_{(0,0)}\left[\mathbf{1}(Y_k=X_k)\alpha^{\#\{k<i<n:Y_i=X_i\}}\middle| Y\right],$$

yielding

$$\tilde{\mathbf{E}}\left[\frac{Z_n}{m^n}\middle| Y\right] \le \frac{1}{m^n} + \frac{\mathbf{E}[\hat{K}-1]}{A}\mathbf{E}_{(0,0)}\left[\sum_{k=0}^{n-1}\mathbf{1}(Y_k=X_k)\alpha^{\#\{k<i<n:Y_i=X_i\}}\middle| Y\right]$$

$$\le 1 + \frac{\mathbf{E}[\hat{K}-1]}{A}\mathbf{E}_{(0,0)}\left[\#\{i\ge 0: Y_i = X_i\}\alpha^{\#\{i\ge 0:Y_i=X_i\}}\middle| Y\right]$$

uniformly in n. The right-hand side is a.s. finite because $\alpha < \alpha_*$. □

Let us note in concluding that in the spatial scenario, the picture is by far less complete than for the non-spatial branching processes discussed in Sections 2 to 4. Even the question of tractable criteria for criticality seems open. Theorem 4 shows that if $\tilde{p}$ is transient and the variance of the mean offspring number is small enough in comparison with a threshold that depends only on the motion, then the classical dichotomy for Galton-Watson processes (without random environments) holds: $m \le 1$ implies almost sure extinction of a single family, while $m > 1$ implies survival with positive probability, and in this case the population grows exponentially.

This naturally leaves us with some questions: First, is the threshold value α_* given in Theorem 4 (c) sharp in the sense that $\alpha > \alpha_*$ would imply $M_\infty = 0$? Arguments from [CSY03] can be adapted to our scenario to show that even for transient $\tilde{p}$ we will have $M_\infty = 0$, if $\mathbf{E}[m_1(Q) \log m_1(Q)]/\mathbf{E}[m_1(Q)]$ is sufficiently large, see Theorem 2.3 (a) there, but there is this a wide gap between the two criteria. In principle, a route to check this would be to directly analyse the distributions of M_n under $\tilde{P}$ (recall that we only checked boundedness of some conditional expectation in the proof of Theorem 4 (c)). This looks like a very hard problem.

Second, even if the growth rate of the population is not captured by m and hence $M_\infty = 0$, this need not necessarily (and in general will not) imply a.s. extinction of a single family: Comparison with the non-spatial case shows at least that $\mathbf{E}[\log(m_1(Q))] > 0$ entails a positive probability of non-extinction irrespective of p.

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