Computing the automorphism group: a concrete example.

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Introduction

In these notes we provide an example of how to compute the automorphism group of a K3 surface knowing its Néron-Severi group. Namely, we will compute the automorphism group of a K3 surface X having the Néron-Severi group isomorphic to the lattice

$$\Gamma_0 = \left(\mathbb{Z}^2, \begin{pmatrix} -2 & 0 \\ 0 & 4 \end{pmatrix}\right).$$

Notice that Γ_0 is a primitive sublattice of $\Lambda_{K3} = (\mathbb{Z}^{22}, U^3 \oplus E_8(-1)^2)$, of rank 2, and with signature $(1,1)$.

A deep result states that for any primitive sublattice $\Gamma \subseteq \Lambda_{K3}$ of signature $(1, \rho - 1)$, with $\rho \leq 20$, there exist an algebraic K3 surface with Néron-Severi group isometric to Γ (see [Mor84, Corollary 1.9]). Furthermore, if $\rho \leq 10$, the primitive embedding $\Gamma \hookrightarrow \Lambda_{K3}$ is unique.

Nevertheless, we will explicitly construct a K3 surface X with Néron-Severi group $\mathrm{NS}(X) \cong \Gamma_0$, and we will see how knowing the geometry of X helps in computing its automorphism group although it is not strictly needed.

In what follows, we will always work over the complex field.

1 Constructing a K3 surface

Let $S \subseteq \mathbb{P}^3(x : y : z : w)$ be an irreducible projective quartic surface with an ordinary double point and no other singularities. Without loss of generality we may assume that the singular point is the point $P = (0 : 0 : 0 : 1)$. It follows that the surface can be written as

$$S \colon Q(x,y,z)w^2 + R(x,y,z)w + G(x,y,z) =: F(x,y,z,w) = 0,$$

where Q, R and G are homogeneous polynomials of degree $2, 3$ and 4 respectively; we will see later that the assumption for P to be an ordinary singular point is equivalent to assuming that the conic $C_2 \subseteq \mathbb{P}^2(x:y:z)$ defined by the equation $Q(x,y,z) = 0$ is smooth.

We also assume the surface S to be general, i.e., it contains no lines. This implies that there is no point $q = (a:b:c) \in \mathbb{P}^2(x:y:z)$ such that $G(q) = R(q) = Q(q) = 0$.

Let $\tilde{S}$ be the blow up of S at the point P, and then consider the strict transform X of S in $\tilde{S}$.

Proposition 1.1. *The surface X is a double cover of $\mathbb{P}^2$ ramified along a smooth sextic curve C_6.*

Proof. Let $B_P\mathbb{P}^3$ be the blow up of $\mathbb{P}^3(x,y,z,w)$ at the point $P = (0:0:0:1)$. Then $B_P\mathbb{P}^3$ is the variety in $\mathbb{P}^3 \times \mathbb{P}^2$ define by

$$\begin{cases} xv = uy \\ xt = uz \\ yt = vz \end{cases},$$

where u, v, t are the coordinates in $\mathbb{P}^2 = \mathbb{P}^2(u:v:t)$. Let $\pi_3 \colon B_P\mathbb{P}^3 \to \mathbb{P}^3$ be the restriction of the canonical projection $\mathbb{P}^3 \times \mathbb{P}^2 \to \mathbb{P}^3$ to $B_P\mathbb{P}^3$. With this notation, we have that the blow up $\tilde{S}$ of S is $\pi_3^{-1}(S)$.

By construction, and since S is irreducible, $\tilde{S}$ has two irreducible components: one is the exceptional divisor $\pi_3^{-1}(P) = \{P\} \times \mathbb{P}^2$, the other is X, the closure of $\pi^3(S - \{P\})$ in $\tilde{S}$.

In order to show that X is a double cover of $\mathbb{P}^2$, consider the open subset $\mathcal{U}_3 \times \mathcal{U}_2$ of $\mathbb{P}^3 \times \mathbb{P}^2$, where

$$\mathcal{U}_3 = \{(x:y:z:w) \in \mathbb{P}^3 \mid w \neq 0\},$$
$$\mathcal{U}_2 = \{(u:v:t) \in \mathbb{P}^2 \mid t \neq 0\}.$$

Notice that $\mathcal{U}_i \cong \mathbb{A}^i$, for $i = 2, 3$, and with an abuse of notation we will use the letters x, y, z and u, v to denote the affine coordinates on $\mathbb{A}^3$ and $\mathbb{A}^2$ respectively.

Using the affine coordinates, we can write $S^0 := \tilde{S} \cap (\mathcal{U}_3 \times \mathcal{U}_2)$ as

$$\begin{cases} F(x,y,z,1) = 0 \\ xv = uy \\ x = uz \\ y = vz \end{cases},$$

from which it follows that S^0 is isomorphic to the surface of $\mathbb{A}^3(u,v,z) \cong B_P\mathbb{P}^3 \cap (\mathcal{U}_3 \times \mathcal{U}_2)$ given by

$$z^2Q(u,v,1) + z^3R(v,u,1) + z^4G(u,v,1) = 0.$$

Notice that S^0 is reducible. The component given by $z = 0$ comes from the exceptional divisor. The other component, given by

$$X^0: Q(u,v,1) + zR(u,v,1) + z^2G(u,v,1) = 0,$$

comes from X, and it is isomorphic to $X \cap (\mathcal{U}_3 \times \mathcal{U}_2)$.

Consider the map $\phi: X^0 \to \mathbb{A}^2$ defined by

$$(u,v,z) \mapsto (u,v).$$

It is easy to see that the map is well defined everywhere, surjective, and $2:1$. It is also easy to see that it ramifies along the sextic plane curve defined by

$$C: R(u,v,1)^2 - 4G(u,v,1)Q(u,v,1) = 0.$$

For different choices of $\mathcal{U}_2$ and $\mathcal{U}_3$, using the same arguments, we get similar maps. Gluing them together we get a double cover $X \to \mathbb{P}^2$ ramified along a curve that is isomorphic to the projective closure C_6 of the sextic C.

Since S has no singularities outside P, the surface X is smooth. Therefore, also C_6 is smooth. □

Theorem 1.2. *The surface X is a K3 surface.*

Proof. See [Huy14, Example 1.1.1.3.iv)]. □

2 Computing the Néron-Severi group

The aim of this section is to compute Néron-Severi group of the surface X we constructed in Section 1. In doing this, we will assume that the surface X is generic, i.e., its Néron-Severi group has rank as small as possible. In particular, we assume that its rank is 2.

Notice that this assumption is not as strong as it might seem. In fact, algebraic K3 surfaces with Néron-Severi group Γ with signature $(1, \rho - 1)$ are parametrised by a dense subset of a complex manifold of dimension $20 - \rho$. The complement of that dense subset parametrises all the algebraic K3 surfaces with picard number strictly larger than ρ, and with a Néron-Severi group admitting Γ as subgroup.

Lemma 2.1. *There is a curve $N \cong \mathbb{P}^1$ on the surface X.*

Proof. As in the proof of Proposition 1.1, let $B_P\mathbb{P}^3$ be the blow up of $\mathbb{P}^3(x,y,z,w)$ at the point $P = (0:0:0:1)$. Then $B_P\mathbb{P}^3$ is the variety in $\mathbb{P}^3 \times \mathbb{P}^2$ defined by

$$\begin{cases} xv = uy \\ xt = uz \\ yt = vz \end{cases},$$

where u, v, t are the coordinates in $\mathbb{P}^2 = \mathbb{P}^2(u:v:t)$. Let $\pi_3: B_P\mathbb{P}^3 \to \mathbb{P}^3$ be the restriction of the canonical projection $\mathbb{P}^3 \times \mathbb{P}^2 \to \mathbb{P}^3$ to $B_P\mathbb{P}^3$. With this notation, we have that the blow up $\tilde{S}$ of S is $\pi_3^{-1}(S)$.

By construction, and since S is irreducible, $\tilde{S}$ has two irreducible components: one is the exceptional divisor $\pi_3^{-1}(P) = \{P\} \times \mathbb{P}^2$, the other is X, the closure of $\pi^3(S - \{P\})$ in $\tilde{S}$.

Consider the open subset $\mathcal{U}_3 \times \mathcal{U}_2$ of $\mathbb{P}^3 \times \mathbb{P}^2$, where

$$\mathcal{U}_3 = \{(x:y:z:w) \in \mathbb{P}^3 \mid w \neq 0\},$$
$$\mathcal{U}_2 = \{(u:v:t) \in \mathbb{P}^2 \mid t \neq 0\},$$

then we have that $X \cap (\mathcal{U}_3 \times \mathcal{U}_2)$ is isomorphic to the surface in $\mathbb{A}^3(u,v,z)$ given by

$$X^0: Q(u,v,1) + zR(u,v,1) + z^2 G(u,v,1) = 0.$$

In this affine patch, the exceptional divisor is the plane given by the equation $z = 0$. It follows that the exceptional divisor intersects X^0 in the curve given by the equation

$$Q(u,v,1) = 0.$$

Recall that Q is an homogeneous polynomial of degree 2.

In this way we have shown the intersection of X with the exceptional divisor in $B_P\mathbb{P}^2$ is

$$N = \{((0:0:0:1),(u,v,t)) \in B_P\mathbb{P}^2 \mid Q(u,v,t) = 0\},$$

that is isomorphic to the planar conic defined by $Q = 0$. By the assumption on the point P to be an ordinary singularity, we have that N is smooth and, therefore, N is isomorphic to $\mathbb{P}^1$. □

Proposition 2.2. *The Néron-Severi group of X is isomoprhic to*

$$\left(\mathbb{Z}^2, \begin{pmatrix} -2 & 0 \\ 0 & 4 \end{pmatrix}\right),$$

and it is spanned by the class of the curve N and by the class H of the hyperplane sections.

Proof. By Lemma 2.1 we have that there is a rational curve N on X. By the adjunction formula it follows that $N^2 = -2$.

Let H be an hyperplane section of X. We can assume that the hyperplane does not pass through P, and so $N \cdot H = 0$. Since outside the exceptional divisor the surface X is isomorphic to S, a projective quartic surface, we have that $H^2 = 4$.

Let Λ the lattice spanned by these two classes,

$$\Lambda = N \cdot \mathbb{Z} \oplus H \cdot \mathbb{Z}.$$

By assumption, $\mathrm{NS}(X)$ has rank 2, so Λ is a sublattice of finite index of the Néron-Severi group $\mathrm{NS}(X)$. Let d be the index of Λ in $\mathrm{NS}(X)$. We have that

$$-8 = \det \Lambda = d^2 \det \mathrm{NS}(X),$$

and so d equals either 1 or 2.

If $d = 1$, then we are done. So assume that $d = 2$. It follows that $\det \mathrm{NS}(X) = -2$. We know that $\mathrm{NS}(X)$ is an even lattice (of rank 2), therefore its intersection matrix is of the form

$$\begin{pmatrix} 2a & c \\ c & 2b \end{pmatrix},$$

and, therefore, $\det \mathrm{NS}(X) = 4ab - c^2$. This means that $\det \mathrm{NS}(X)$ can be either odd or divisible by 4; in particular, it cannot be equal to -2. This concludes the proof. □

Corollary 2.3. *On the surface X there are at least two curves isomorphic to $\mathbb{P}^1$, namely N and N'. The divisor of the curve N' is linearly equivalent to the divisor $-3N + 2H$.*

Proof. By Lemma 2.1 we know that N is a rational curve on X. By Proposition 1.1, we know that X is a double cover of $\mathbb{P}^2$ ramified along a smooth sextic curve. Let ι be the involution of X swapping the elements in the fibers of the projection $\phi: X \to \mathbb{P}^2$. Since the projection is ramified along a smooth sextic we have that N is not contained in the branch locus, and therefore the involution ι sends N to another rational curve, say N'.

Our aim is now to write N' in terms of N and H. Recall that N is sent by ϕ to the plane conic C_2 defined by $Q = 0$. The class of this conic is linearly equivalent to $2L$ in $\mathrm{Pic}(\mathbb{P}^2)$, where L is the class of a line. Then, we have that the pullback of the conic C_2 in $\mathbb{P}^2$ is

$$\phi^* 2L = N + N' \in \mathrm{NS}(X).$$

On the plane we have that $L^2 = 1$, and then, since ϕ is $2:1$, we have that $(\phi^* L)^2 = 2$. It follows that

$$(N + N')^2 = (\phi^* 2L)^2 = (2\phi^* L)^2 = 4(\phi^* L)^2 = 8.$$

Using this equality, we get

$$8 = (N + N')^2 = -2 - 2 + 2N \cdot N' = -4 + 2N \cdot N',$$

and, therefore,

$$N \cdot N' = 6.$$

Since N and H span the Néron-Severi group, we have that N' can be written in the form $N' = aN + bH$, for some $a, b \in \mathbb{Z}$. The condition that $N \cdot N' = 6$ yields that $a = -3$. Then $N' = -3N + bH$. Since N' is a rational curve we have that $N'^2 = -18 + 4b^2 = -2$, and hence $b = \pm 2$. To compute the correct sign, consider the class hyperplane section H: since N' is an actual curve on X, we have that the intersection number $H \cdot N'$ must be positive. So we have:

$$H \cdot N' = H \cdot (-3N + bH) = 4b > 0,$$

and, therefore, b must be positive. We can finally conclude that $N' = -3N + 2H$. □

Remark 2.4. We will see later (cf. Lemma 3.9), that N and N' are infact the only two -2-curves on X.

Remark 2.5. In Proposition 1.1 we show that X is a double cover of $\mathbb{P}^2$ ramified along a sextic. Using the description of the Néron-Severi group (i.e., the Picard group of X) given in Proposition 2.2, it turns out that the $2:1$ cover $X \to \mathbb{P}^2$ is the map given by the linear system $|H - N|$.

3 Computing the ample cone

Given the Néron-Severi group isomorphic to the lattice $\Gamma_0 = \left(\mathbb{Z}^2, \begin{pmatrix} -2 & 0 \\ 0 & 4 \end{pmatrix}\right)$, in this section we compute the ample cone of X in two different ways (cf. Proposition 3.5). First, we will compute the ample cone of X using very little of the theory developed in the previous sections, namely, we will only assume that the divisor $H - N$, which we know to be ample by Remark 2.5. Notice that this assumption is needed because a priori the ample cone is note uniquely determined: in fact the Weyl group acts transitively on the set of chambers of the Néron-Severi group (see [Huy14, Proposition 8.2.6]). In the second proof, we will use all the information coming from the previous sections, making the proof much shorter.

Let $\{N, H\}$ be a basis for $\mathrm{NS}(X)$, with $N^2 = -2$, $H^2 = 4$, and $H \cdot N = 0$. Let $D = xN + yH$ be an element of $\mathrm{NS}(X)$. Then $D^2 > 0$ if and only if $x^2 - 2y^2 < 0$. If we identify $\mathrm{NS}_{\mathbb{R}}(X)$ with the affine plane $\mathbb{A}^2_{\mathbb{R}}(x, y)$, this means that $D^2 > 0$ if and only if D lies in the region

$$\mathcal{C}_X = \{(x, y) \mid x^2 - 2y^2 < 0\}.$$

We have already seen, but it is also easy to check directly, that $\mathcal{C}_X$ has two connected components, one of those containing the ample divisor. In fact,

$$\mathcal{C}_X = (\mathcal{C}_X \cap \{y < 0\}) \cup (\mathcal{C}_X \cap \{y > 0\}).$$

If we assume that $H - N$ is ample, then

$$\mathcal{C}_X^+ = \mathcal{C}_X \cap \{y > 0\}.$$

By the Nakai-Moishezon theorem we have that a divisor $E \in \mathrm{NS}(X)$ is ample if and only if $E \in \mathcal{C}_X$ and $E \cdot \delta > 0$ for every δ in

$$\Delta^+ := \{\delta \in \mathrm{NS}(X) \mid \delta^2 = -2,\ \delta > 0\}.$$

Remark 3.1. Recall that, by the Riemann-Roch theorem, if δ is a -2-class, then either δ or $-\delta$ is effective.

Remark 3.2. Using the Nakai-Moishezon criterion and the assumption for $H - N$ to be ample, it follows that N is effective. Infact $N^2 = -2$, and so either N or $-N$ is effective. By the Nakai-Moishezon criterion, the intersection of an ample divisor with an effective divisor is positive, while $(H - N) \cdot (-N) = -2 < 0$. Therefore, N is effective.

Remark 3.3. Actually, by Lemma 2.1, we *know* that N is effective. In fact we know that N is the class of a rational curve. The same holds for the divisor $N' = -3N + 2H$ (cf. Corollary 2.3).

We then interested in knowing more about the effective -2-classes of $\mathrm{NS}(X)$.

Lemma 3.4. *There are infinitely many -2-classes in $\mathrm{NS}(X)$.*
Namely, $D \in \mathrm{NS}(X)$ is a -2-class if and only D is of the form

$$nN + hH,$$

where $n, h \in \mathbb{Z}$ are such that

$$\pm(1 + \sqrt{2})^{2k} = n + h\sqrt{2},$$

for some integer k.

Proof. Let D be divisor in $\mathrm{NS}(X)$, then D can be written as $D = nN + hH$, with $n, h \in \mathbb{Z}$.

Saying that D is a -2-class is equivalent to saying that $D^2 = -2n^2 + 4h^2 = -2$, or, equivalently, that

$$n^2 - 2h^2 = 1.$$

Now notice that $n^2 - 2h^2$ is the norm of the element $n + h\sqrt{2}$ in $\mathbb{Z}[\sqrt{2}]$. Then saying that D is a -2-class is equivalent to saying that the element $n + h\sqrt{2}$ is a unit in $\mathbb{Z}[\sqrt{2}]$ with norm equal to 1. It is a well known fact that the units in $\mathbb{Z}[\sqrt{2}]$ are the powers of $1 + \sqrt{2}$, up to sign; in particular, the units with norm 1 are, up to sign, the even powers of $1 + \sqrt{2}$. This concludes the proof. □

Proposition 3.5. *The ample cone of the surface X is the following set:*

$$\mathrm{Amp}(X) = \{xN + yH \in \mathrm{NS}(X) \mid x < 0,\ y > -3/4x\}.$$

We give two proves of this proposition: the first one is independent of the first two sections, i.e., it is independent of the geometry of the surface X but it use the assumption that the divisor $H - N$ is ample; the second proof relies on Sections 1 and 2, in particular on the fact that, by construction, we have that the classes N and N' are classes of actual rational curves on X.

Naive proof. Recall that the ample cone is the chamber of the Néron-Severi group contained in the effective cone $\mathcal{C}_X^+$ and containing one (and therefore all) the ample divisors. Using the assumption for the divisor $H - N$ to be ample, all we need to do is to determine the chamber it is in.

Let us identify the vector space $\mathrm{NS}(X)_\mathbb{R}$ with the affine plane $\mathbb{A}^2(x, y)$. It is easy to see that the divisors N and $N' = -3N + 2H$ are both -2-divisors and they determine two walls, represented by the lines with equations

$$x = 0 \quad \text{and} \quad 4y + 3x = 0$$

respectively.

Let Σ be the region of $\mathcal{C}_X^+$ determined by these two lines, namely

$$\Sigma := \{(x, y) \in \mathbb{A}^2 \mid x < 0,\ y > -3/4x\}.$$

Notice that $H - N \in \Sigma$. We claim that Σ is a chamber.

In order to show this we need to show that no other walls pass through Σ. By Lemma 3.4 we have that all the walls are represented by lines of the form:

$$H_k\colon n(k)x - 2h(k)y = 0,$$

where $n(k), h(k) \in \mathbb{Z}$ are the integers such that

$$\pm(1+\sqrt{2})^{2k} = n(k) + h(k)\sqrt{2},$$

for some integer k. It is easy to see that for $k = 0, 1$ we recover the walls described by N and N'; for $k > 0$, we have $n(k), h(k) > 0$ and so the line H_k is a line that passes through the origin with a positive slope, therefore it cannot pass through Σ. One can show that for $k < 0$, the quantity $\frac{n(k)}{2h(k)}$ is always greater than $-3/4$ (and in fact that it goes to $-\frac{1}{\sqrt{2}}$ for k that goes to infinity). This shows that Σ is a chamber, and an ample divisor lies in it. Therefore, Σ is the ample chamber.

□

Before giving the geometric proof, we first show that in fact any K3 surface with Néron-Severi group isomorphic to Γ_0 is a double cover of $\mathbb{P}^2$ and then we recall some general results about the ample cone of a K3 surface.

Proposition 3.6. *Let Y be a projective K3 surface with $\mathrm{NS}(Y) \cong \Gamma_0$. Then Y is isomorphic to a double cover of $\mathbb{P}^2$ ramified along smooth sextic.*

Proof. Let N, H be the generators of $\mathrm{NS}(Y)$, with $N^2 = -2$ and $H^2 = 4$. Consider then the divisor $D = H - N$ and notice that $D^2 = 2 > 0$. Without loss of generality we can assume that D is ample. The linear system $|D|$ induces a rational map

$$\psi_D : Y \dashrightarrow \mathbb{P}^d,$$

where $d = h^0(D) - 1$. From the Riemann-Roch for surfaces, applying Serre's duality and keeping in mind that Y is a K3 surface, and therefore it has trivial canonical divisor, we have the following equality:

$$h^0(D) - h^1(D) + h^0(-D) = D^2/2 + 2. \tag{1}$$

Since D is ample, then $h^0(-D) = 0$, and $h^1(D) = 0$ (see [Huy14, Proposition 2.3.1]). Then the (1) yields that $h^0(D) = 3$. Let x, y, z be a basis for $h^0(D)$. Then the map ψ_D is the rational map from Y to $\mathbb{P}^2$ defined by

$$P \mapsto (x(P) : y(P) : z(P)).$$

Again applying the Riemann-Roch to $2D$ we get that $h^0(2D) = 6$ and a basis is given by exactly all the possible monomials of degree 2 in x, y, z. Using the same argument, we get that $h^0(3D) = 11$, and so a basis for $h^0(3D)$ is given by all the possible monomials of degree 3 in x, y, z (there are ten of them) plus one extra element, say w. Iterating this argument for tD, with

$t = 4, 5, 6$ one finds that $h^(6D) = 38$ while the monomials of weighted degree 6 in x, y, z, w are 39. This means that we have an injection $Y \hookrightarrow \mathbb{P}(1,1,1,2)$ whose image is isomorphic to a surface given by an equation of the form

$$w^2 = F(x, y, z),$$

where F is a homogeneous polynomial of degree 3. This shows that Y is a double cover of $\mathbb{P}^2$ ramified along the sextic given by $F = 0$. Since Y is a K3 surface, hence it is smooth, also the ramification locus must be smooth. This completes the proof. □

Theorem 3.7. *Let Y be a projective K3 surface. Then*

$$\mathrm{Amp}(Y) = \{D \in \mathcal{C}_Y^+ \mid \forall\, \delta \cong \mathbb{P}^1,\ D \cdot \delta > 0\}.$$

Proof. See [Huy14, Corollary 8.1.7]. □

Proposition 3.8. *Let Y be a projective K3 surface. Then every class of a rational curve of Y defines a wall of the ample chamber* $\mathrm{Amp}(Y)$*.*

Proof. See [Huy14, Remark 8.2.8]. □

Corollary 3.9. *Let Y be a projective K3 surface. Assume Y has Picard number equal to* 2*. Then there are at most 2 rational curves on Y.*

Proof. If the Picard number of Y is 2, then we can identify $\mathrm{NS}_{\mathbb{R}}(Y)$ with the affine plane. The ample chamber $\mathrm{Amp}(Y)$ is a cone contained in the positive cone $\mathcal{C}_Y^+$. Notice that the vertex of both cones is the origin. Therefore, the cone $\mathrm{Amp}(Y)$ is determined by at most 2 lines passing through the origin other than the external rays of $\mathcal{C}_Y^+$. By Proposition 3.8, every rational curve on Y determines a wall. The statement follows. □

Geometric proof of 3.5. By Proposition 3.6 we know that X is a double cover of $\mathbb{P}^2$, amd therefore we can apply the theory developed in Sections 1 and 2. By Corollary 2.3 we have that on X there are at least two rational curves, N and N'. By Corollary 3.9 it follows that these are the only two. Then the statement follows from Theorem 3.7. □

4 Lattices' interlude

In this section we quickly recall some basic facts about lattices we will need in the following section.

We define a lattice to be a pair (Λ, b) consisting of a free $\mathbb{Z}$-module of finite rank Λ and a symmetric bilinear form

$$b\colon \Lambda \times \Lambda \to \mathbb{Z}\,,$$

called the intersection form of the lattice.

Let Λ be a lattice, then we define the dual of Λ to be

$$\Lambda^* := \{x \in \Lambda_{\mathbb{Q}} = \Lambda \otimes \mathbb{Q} \mid \forall\, \lambda \in \Lambda,\ x \cdot \lambda \in \mathbb{Z}\} \cong \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z}).$$

We define the discriminant group of Λ to be the quotient group

$$A_\Lambda := \Lambda^*/\Lambda.$$

Lemma 4.1. *Let Λ be an unimodular lattice, $\Gamma \subseteq \Lambda$ a primitive sublattice and let $\Gamma^\perp$ denote the orthogonal complement of Γ in Λ. Then*

$$A_\Gamma,\ A_{\Gamma^\perp} \cong \Lambda/\Gamma \oplus \Gamma^\perp,$$

and, in particular,

$$A_\Gamma = A_{\Gamma^\perp}$$

(here the equality means that there is a natural isomorphism).

Proof. See Lemma [BHPVdV04, Lemma I.2.5]. □

Lemma 4.2. *Let Λ be a unimodular lattice, $\Gamma \subseteq \Lambda$ a primitive sublattice and let $\Gamma^\perp$ denote the orthogonal complement of Γ in Λ. Let $\mathcal{O}(\Lambda)_\Gamma$ denote the set of isometries of Λ sending Γ to itself.*

Let $\varrho\colon \mathcal{O}(\Lambda)_\Gamma \to \mathcal{O}(\Gamma) \times \mathcal{O}(\Gamma^\perp)$ defined by

$$\alpha \mapsto (\alpha_{|\Gamma}, \alpha_{|\Gamma^\perp}).$$

Then the map ϱ is well defined, injective, and the image is given by

$$\{(\beta, \beta_\perp) \in \mathcal{O}(\Gamma) \times \mathcal{O}(\Gamma^\perp) \mid \bar{\beta} = \bar{\beta}_\perp\} =: R\,,$$

where $\bar{\beta}$ and $\bar{\beta}_\perp$ denote the maps induced by β and $\beta_\perp$ respectively on the discriminant groups A_Γ and $A_{\Gamma'}$.

Remark 4.3. Notice that the equality $\bar{\beta} = \bar{\beta}_\perp$ only makes sense if we identify A_Γ and $A_{\Gamma^\perp}$ (cf. Lemma 4.1).

Proof of Lemma 4.2. Notice that, since Λ is unimodular, the dual lattice Λ^* is isomorphic to Λ and we have an inclusion of abelian groups (with finite indexes)

$$\Gamma \oplus \Gamma^\perp \hookrightarrow \Lambda \hookrightarrow \Gamma^* \oplus (\Gamma^\perp)^*.$$

First we show that the map ϱ is well defined. Let α be an element of $\mathcal{O}(\Lambda)_\Gamma$. Then, by definition, the restriction $\alpha_{|\Gamma}$ is an element of $\mathcal{O}(\Gamma)$; since α is an isometry, it sends $\Gamma^\perp$ to itself.

To show that ϱ is injective, consider two elements α, β in $\mathcal{O}(\Lambda)_\Gamma$ such that

$$(\alpha_{|\Gamma}, \alpha_{|\Gamma^\perp}) = (\beta_{|\Gamma}, \beta_{|\Gamma^\perp}).$$

The $\mathbb{Q}$-linear extension of $\alpha_{|\Gamma}, \alpha_{|\Gamma^\perp}$ and $\beta_{|\Gamma}, \beta_{|\Gamma^\perp}$ induce two maps, α' and β' on $\Gamma^* \oplus (\Gamma^\perp)^*$ such that $\alpha'_{|\Lambda} = \alpha$ and $\beta'_{|\Lambda} = \beta$. But from the assumption on α and β we have that $\alpha' = \beta'$. The statement follows.

Finally, we want to show that the image of ϱ is exactly R. It is trivial to see that it is contained. To show the other inclusion, consider a pair $(\beta, \beta_\perp)$ in R. Then, as before, the $\mathbb{Q}$-linear extensions β' and $\beta'_\perp$ of β and $\beta_\perp$ to Γ^* and $(\Gamma^\perp)^*$ respectively, define an isometry α of $\Gamma^* \oplus (\Gamma^\perp)^*$ such that $\alpha_{|\Gamma} = \beta$ and $\alpha_{|\Gamma^\perp} = \beta_\perp$. We only need to show that α sends Λ to itself. Observe that if $\bar{\alpha}$ denotes the map induced by α on $(\Gamma^* \oplus (\Gamma^\perp)^*)/(\Gamma \oplus \Gamma^\perp)$, then saying that α sends Λ to itself is equivalent to saying that $\bar{\alpha}$ sends $\Lambda/(\Gamma \oplus \Gamma^\perp)$ to itself. But, by Lemma 4.1, $\bar{\alpha}(\Lambda/(\Gamma \oplus \Gamma^\perp)) = \bar{\beta}'(A_\Gamma) \subseteq A_\Gamma = \Lambda/(\Gamma \oplus \Gamma^\perp)$. This concludes the proof. □

5 Computing the automorphism group

Our goal in this section is to compute the automorphism group of X using the information we gathered about the ample cone. As in Section 2, in order to fulfill our goal we will need to assume that our surface X is 'general enough' (cf. Remark 5.6).

Proposition 3.5 describes the ample cone of our surface X, and in particular it says that the walls of the ample cone are the walls given by the divisors N and N'.

This description is very useful in order to apply the Torelli theorem. Let us recall it.

Theorem 5.1. *Let Y, Y' be K3 surfaces, and let $\Phi\colon H^2(Y, \mathbb{Z}) \to H^2(Y', Z)$ be an effective Hodge isometry. Then there is a unique isomorphism $f\colon Y' \to Y$ such that $\Phi = f^*$.*

Proof. See [BHPVdV04, Theorem VIII.11]. □

Recall that an effective Hodge isometry is an isomorphism of abelian group $\Phi: H^2(Y,\mathbb{Z}) \to H^2(Y',\mathbb{Z})$ such that

1. it preserves the intersection form;

2. it preserves the Hodge structure;

3. it preserves the ample cone.

For $Y = Y'$, we have the following easy corollary of Theorem 5.1.

Corollary 5.2. *Let Y be a K3 surface, and let $\Phi: H^2(Y,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$ be an effective Hodge isometry. Then there is a unique automorphism $f: Y \to Y$ such that $\Phi = f^*$.*

Corollary 5.2 tells us that an eventual automorphism f of X would induce an effective Hodge isometry $\Phi: H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{Z})$. The restriction $\varphi = \Phi_{|\mathrm{NS}(X)}: \mathrm{NS}(X) \to \mathrm{NS}(X)$ would hence be an isometry of $\mathrm{NS}(X)$.

Lemma 5.3. *There are exactly two isometries of* $\mathrm{NS}(X)$ *that preserve the ample cone: the identity* $\mathrm{id}_{\mathrm{NS}(X)}: \mathrm{NS}(X) \to \mathrm{NS}(X)$ *and the map*

$$\varphi: \mathrm{NS}(X) \to \mathrm{NS}(X),\ xN + yH \mapsto -(3x+4y)N + (2x+3y)H.$$

Proof. Notice that the map φ is an isometry of $\mathrm{NS}(X)$ preserving the ample cone and swapping its extremal rays, that is, $\varphi(N) = N'$ and $\varphi(N') = N$.

Let σ be an isometry of $\mathrm{NS}(X)$ sending ample divisors to ample divisors, i.e., σ fixes the ample cone. Then σ either fixes the extremal rays of the cone, or it swaps them. This means that either $\sigma = \pm\mathrm{id}_{\mathrm{NS}(X)}$ or $\sigma = \pm\varphi$. The condition on σ to send ample divisors to ample divisors gives the right choice of the sign. □

The maps $\mathrm{id}_{\mathrm{NS}(X)}$ and φ are isometries of $\mathrm{NS}(X)$ preserving the ample cone, but in order to apply Corollary 5.2 we need to know whether they come from an isometry of $H^2(X,\mathbb{Z})$ or not.

Recall that the Néron-Severi group $\mathrm{NS}(X)$ is a primitive sublattice of the unimodular lattice $H^2(X,\mathbb{Z}) \cong \Lambda_{K3}$, where $\Lambda_{K3} = (\mathbb{Z}^{22}, U^3 \oplus E_8(-1)^2)$. Let $T(X)$ denote the *transcendental lattice* of X, that is, the orthogonal complement $\mathrm{NS}(X)^\perp$ inside $H^2(X,\mathbb{Z})$.

Lemma 5.4. *The discriminant group $A_{\mathrm{NS}(X)}$ (or, equivalently, $A_{T(X)}$) is isomorphic to the group*

$$\frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{4\mathbb{Z}}.$$

Proof. Consider the embedding $\mathrm{NS}(X) \hookrightarrow \Lambda_{K3}$ given by

$$N \mapsto (1, -1, 0, ..., 0), \tag{2}$$

$$H \mapsto (0, 0, 1, 2, 0, ..., 0). \tag{3}$$

Using the embedding defined above, we have that a divisor $D = (aN + bH) \in \mathrm{NS}(X)$ is sent to the vector $(a, -a, b, 2b, 0, ..., 0)$. Therefore the intersection number $x \cdot D$ is the quantity

$$-2ap + 4bq,$$

from which we have that

$$\mathrm{NS}(X)^* \cong \mathbb{Z}\left[\frac{1}{2}\right] \times \mathbb{Z}\left[\frac{1}{4}\right]$$

via the isomorphism

$$(p, -p, q, 2q, 0, ..., 0) \mapsto (p, q).$$

Therefore the discriminant group $A_{\mathrm{NS}(X)}$ is isomorphic to the group

$$\frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{4\mathbb{Z}}.$$

□

Proposition 5.5. *Let $\mathcal{O}(T(X))$ denote the group of the Hodge isometries of the transcendental lattice $T(X)$. Then $\mathcal{O}(T(X))$ is a nontrivial cyclic group of order a power of two.*

Proof. First of all, $\pm\mathrm{id}_{T(X)}$ are trivially Hodge isometry of $T(X)$.

We have seen that the group of Hodge isometries of $T(X)$ is a finite cyclic group, isomoprhic to the μ_n, the group of n-th roots of unity, for some n (see [Huy14, Corollary 3.4.4]).

Let ζ be an element of μ_n. By an abuse of notation, we denote by ζ also the corresponding Hodge isometry of $T(X)$.

By definition we know that $\zeta^n = 1$, therefore the order of ζ divides n.

Let $p \neq 2$ be a prime dividing n. Then a suitable power of ζ, say σ, has order p.

The eigenvalues of σ are thus p-th roots of unity, not all equal to 1. In fact, if σ has an eigenvalue equal to 1 then either the sublattice $T(X)^\sigma_{\mathbb{C}}$ or its orthogonal complement (in $T(X)$) would properly contain $H^{2,0}(X) \oplus H^{0,2}(X)$, i.e., either the orthogonal complement in $T(X)$ of $T(X)^\sigma$ or $T(X)^\sigma$ itself would be properly contained in $\mathrm{NS}(X)$. This is a contradiction since

$T(X)$ is the orthogonal complement of $\mathrm{NS}(X)$ in $H^2(X,\mathbb{Z})$. It follows that τ has no eigenvalues equal to 1 and therefore

$$\sigma^{p-1} + \sigma^{p-2} + \ldots + 1 = 0.$$

The isometry σ induces an isometry of order p on $A_{T(X)} \cong \frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{4\mathbb{Z}}$. The automorphism group of $\frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{4\mathbb{Z}}$ is isomorphic to dihedral group of order 8, that has no elements of order p. It follows that σ induces the identity on $A_{T(X)}$. But then, for $x = (1,0) \in A_{T(X)}$ we have

$$0 = (\sigma^{p-1} + \sigma^{p-2} + \ldots + 1)x = px = x,$$

since $p \equiv 1(2)$. This is a contradiction. Hence we have shown that n must be a power of 2.

□

Proposition 5.5 tells us that the Hodge isometries of the transcendental lattice form a cyclic group of order a power of 2 but in fact, by assuming the surface X to be general enough, it follows that the only Hodge isometries of $T(X)$ are $\pm\mathrm{id}_{T(X)}$ (cf. Remark 5.6).

Remark 5.6. Let Y be a generic K3 surface. Then $\pm\mathrm{id}_{T(Y)}$ are the only Hodge isometries of the transcendental lattice T of Y.

To see this, let T be a fixed sublattice of the K3 lattice Λ_{K3}, let $(2, t-2)$ be its signature.

Also, let ω be a generator for $T^{2,0} = \mathbb{C}\cdot\omega \subseteq T \otimes \mathbb{C}$ such that $\omega^2 = 0$ and $\omega \cdot \bar{\omega} > 0$. These data determine a K3 surface Y with transcendental lattice $T_Y \subseteq T$ (via the period map).

If for any $Y \in T$ we have that ω does not lie in $Y^{\perp}$, then $T_Y = T$. Assume this is the case.

If f is an automorphism of Y, then f^* maps $T^{2,0}$ to itself. But $T^{2,0}$ is one dimensional, therefore it is an eigenspace of f^*.

This means that to be sure that all the Hodge isometries of T are just $\pm$id all we need to require is that $T^{2,0}$ is not contained in the eigenspace of ϕ, for any isometry ϕ of T.

The group of isometries of T is a countable set, since $T = \mathbb{Z}^t$ and an isometry of it is given by a matrix with integer coefficients.

Then we will have that the Hodge isometries of T are just $\pm id$ if ω lies outside a countable union of complex subspaces of $T \otimes \mathbb{C}$, the union of the eigenspaces of the automorphisms of T.

Proposition 5.7. *The isometries* $\mathrm{id}_{\mathrm{NS}(X)}$ *and* φ *of* $\mathrm{NS}(X)$ *extend to* id_{H^2} *and* Φ*, effective Hodge isometries of* $H^2(X,\mathbb{Z})$*. Furthermore, these two are the only effective Hodge isometries of* $H^2(X,\mathbb{Z})$*.*

Proof. Lemma 5.3 tells us that there are exactly two Hodge isometries of the Néron-Severi group, namely the identity $\mathrm{id}_{\mathrm{NS}(X)}$ and φ; Remark 5.6 tells us that $\pm\mathrm{id}_{T(X)}$ are the only Hodge isometries of the transcendental lattice $T(X)$.

It is trivial that both $\mathrm{id}_{NS(X)}$ and $\mathrm{id}_{T(X)}$ induce the identity on the discriminant lattice $A_X = \mathrm{NS}(X)^*/\mathrm{NS}(X) = T(X)^*/T(X)$; it is also trivial that $-\mathrm{id}_{T(X)}$ induces $-id$ on A_X; finally, it is easy to see that φ induces $-id$ on A_X as well.

Then, by Lemma 4.2, it follows that the pairs $(\mathrm{id}_{\mathrm{NS}(X)}, \mathrm{id}_{T(X)}), (\varphi, -\mathrm{id}_{T(X)})$, and only these two pairs, extend to two Hodge isometries of Λ_{K3}.

Then we have exactly two Hodge isometries of $\Lambda_{K3} = H^2(X, \mathbb{Z})$, say the identity and Φ. □

Theorem 5.8. *The K3 surface X has automorphic group*

$$\mathrm{Aut}(X) = \{\mathrm{id}_X, \iota\} \cong \frac{\mathbb{Z}}{2\mathbb{Z}}.$$

Proof. By Proposition 5.7 we know that we only have two effective Hodge isometries, $\{id_{H^2}, \Phi\}$. By Corollary 5.2 we know that the effective Hodge isoemtries are in a $1:1$ correspondence with the automorphisms of X, the correspondence being given by $f \mapsto f^*$, where f is an automorphism of X. Obviously id_{H^2} corresponds to the identity id_X. The isometry Φ correspond to another nontrivial automorphism ι, that will be an involution of X. □

Remark 5.9. Considering the geometric information coming from the first two sections, we already knew that we had such two automorphisms, being X a double cover of the plane. Theorem 5.8 shows that these two are the only ones.

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