Density of Rational Points on a Family of Diagonal Quartic Surfaces

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A family of diagonal quartic surfaces

For any $c_1, c_2 \in \mathbb{Q}^\times$, let $W_{c_1,c_2} = W$ be the surface given by

$$(x^2 - 2c_1y^2)(x^2 + 2c_1y^2) = c_2(z^2 + 2zw + 2w^2)(z^2 - 2zw + 2w^2).$$

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Theorem

Let c_1, c_2 and W be as above.
Let $P = (x_0 : y_0 : z_0 : w_0)$ be a rational point on W with x_0 and y_0 both nonzero.
If $|2c_1|$ is a square in $\mathbb{Q}^\times$, then also assume that z_0, w_0 are not both zero.
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Then the set of rational points on the surface is Zariski dense.

Fibrations from W to $\mathbb{P}^1$:

$$\psi_1 \colon (x : y : z : w) \mapsto (x^2 - 2c_1y^2 : z^2 + 2zw + 2w^2) = (c_2(z^2 - 2zw + 2w^2) : x^2 + 2c_1y^2),$$

$$\psi_2 \colon (x : y : z : w) \mapsto (x^2 - 2c_1y^2 : z^2 - 2zw + 2w^2) = (c_2(z^2 + 2zw + 2w^2) : x^2 + 2c_1y^2).$$

Elliptic fibers

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The fiber F of ψ_1 above $(s : 1)$, with $s \in \overline{\mathbb{Q}}$, is given by:

$$\begin{cases} x^2 - 2c_1 y^2 & = s(z^2 + 2zw + 2w^2) \\ c_2(z^2 - 2zw + 2w^2) & = s(x^2 + 2c_1 y^2) \end{cases}$$

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Most fibers have genus 1: intersection of two smooth quadrics in $\mathbb{P}^3$.
The Jacobian of the fiber is isomorphic over $\mathbb{Q}$ to the elliptic curve given by:

$$\boxed{y^2 = x^3 + 4c_1(c_2^2 - s^4)x^2 + 4c_1^2(c_2^4 - 34s^4c_2^2 + s^8)x.}$$

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Its j-invariant and discriminant are

$$j = \frac{2(s^8 + 94s^4c_2^2 + c_2^4)^3}{c_2^2s^4(s^4 - 6s^2c_2 + c_2^2)^2(s^4 + 6s^2c_2 + c_2^2)^2},$$

$$d = 2^{17}s^4c_1^6c_2^4(s^4 - 6s^2c_2 + c_2^2)^2(s^4 + 6s^2c_2 + c_2^2)^2.$$

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Remark: If we assume that on the fiber there is at least one rational point then the fiber is isomorphic to its Jacobian.

Elliptic fibers

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We have exactly ten singular fibers, for both ψ_1 and ψ_2. Namely the fibers above $(1:0), (0:1), (s:1)$ with

$$s \in S := \{(\pm 1 \pm \sqrt{2})\gamma_2^2, i(\pm 1 \pm \sqrt{2})\gamma_2^2\},$$

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Question 2: Are there rational points on the singular fibers?

Not above the points $(s:1)$, with s in S. There may be rational points on the fibers of ψ_1 and ψ_2 above the points $(0:1)$ and $(1:0)$, whose union is given by:

$$\begin{cases} x^4 - 4c_1^2 y^4 & = 0 \\ z^4 + 4w^4 & = 0 \end{cases}$$

Elliptic fibers

Proposition 2.2.3

Let W and ψ_1 defined as before.
If $|2c_1|$ is not a square in $\mathbb{Q}$ then there are no rational points on any singular fiber.
If $2c_1$ is a square in $\mathbb{Q}$ then there are exactly two rational points on the fiber above $(0:1)$ and this is the only singular fiber with rational points.
If $-2c_1$ is a square in $\mathbb{Q}$ then there are exactly two rational points on the fiber above $(1:0)$ and this is the only singular fiber with rational points.

The same holds for ψ_2.

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The same holds for ψ_2.

Lemma 2.2.4

The intersection of the fibers above $(0:1)$ and $(1:0)$ are the following:

$$\begin{aligned} F_0 \cap G_0 &= \{(\pm\sqrt{2c_1} : 1 : 0 : 0)\}, \\ F_0 \cap G_\infty &= \{(0:0:\sqrt{2}\zeta_8 : 1), (0:0:-\sqrt{2}\zeta_8^3 : 1)\}, \\ F_\infty \cap G_0 &= \{(0:0:-\sqrt{2}\zeta_8 : 1), (0:0:\sqrt{2}\zeta_8^3 : 1)\}, \\ F_\infty \cap G_\infty &= \{(\pm\sqrt{-2c_1} : 1 : 0 : 0)\}. \end{aligned}$$

2-torsion points

Any smooth fiber with at least one rational point, say $P = (x_0 : y_0 : z_0 : w_0)$, is isomorphic over the rationals to the elliptic curve

$$\boxed{y^2 = x^3 + 4c_1(c_2^2 - s^4)x^2 + 4c_1^2(c_2^4 - 34s^4c_2^2 + s^8)x.}$$

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Rational 2-torsion points on the fiber	**Non rational 2-torsion points on the fiber**
$P = (x_0 : y_0 : z_0 : w_0)$ $T_0 = (-x_0 : -y_0 : z_0 : w_0)$	$T_1 = (-\sqrt{2}x_0 : \sqrt{2}y_0 : 2w_0 : z_0)$ $T_2 = (\sqrt{2}x_0 : -\sqrt{2}y_0 : 2w_0 : z_0)$

4-torsion points

Claim (Theorem 3.1.4):

Let F be a smooth fiber of ψ_1 with a rational point P on it. Then (F,P), viewed as an elliptic curve, has no rational nontrivial 4-torsion points.

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Recall that (F, P) is isomorphic over the rationals to the elliptic curve

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The 4-division polynomial of the elliptic curve above is

$$\boxed{f_4(x) = 8xh_1(x)h_2(x)h_3(x),}$$

where

$$\begin{aligned}
h_1(x) =& x^2 + 4c_1(c_2^2 - s^4)x + 4c_1^2(c_2^4 - 34s^4c_2^2 + s^8),\\
h_2(x) =& x^2 - 4c_1^2(s^8 - 34s^4c_2^2 + c_2^4),\\
h_3(x) =& x^4 + 8c_1(c_2^2 - s^4)x^3 + 24c_1^2(s^8 - 34s^4c_2^2 + c_2^4)x^2 +\\
& + 32c_1^3(-s^{12} + 35s^8c_2^2 - 35s^4c_2^4 + 32c_2^6)x +\\
& + 16c_1^4(s^{16} - 68s^{12}c_2^2 + 1158s^8c_2^4 - 68s^4c_2^6 - c_2^8).
\end{aligned}$$

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For any $s \in \mathbb{Q}^\times$ the polynomial $h_2(x) = x^2 - 4c_1^2(s^8 - 34s^4c_2^2 + c_2^4)$ has no rational roots.

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Finding a rational s such that h_2 admits a rational root is equivalent to finding a rationals point on the surface $D \subset \mathcal{U} := \mathbb{A}^3(z, s, c_2) \cap \{sc_2 \neq 0\}$ given by:

$$D\colon z^2 - (s^8 - 34s^4c_2^2 + c_2^4) = 0.$$

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Using the map from D to $\mathbb{P}^2$ sending

$$(z, s, c_2) \mapsto (z/c_2 : s^2 : c_2),$$

one can see that the existence of rational points on D implies the existence of rational points on the curve $\overline{C} \subset \mathbb{P}^2(X, Y, Z)$ defined by:

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Notice that on $\overline{C}$ there are at least three rational points:

$$(1 : 0 : 1), (1 : 0 : -1), (1 : 0 : 0).$$

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Notice that on $\overline{C}$ there are at least three rational points:

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The desingularization of the curve $\overline{C}$ is isomorphic over $\mathbb{Q}$ to the elliptic curve

$$\boxed{H\colon y^2 = x^3 - x^2 - 24x - 36.}$$

$$H(\mathbb{Q}) \cong \mathbb{Z}/4\mathbb{Z}$$

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Let F be a smooth fiber of ψ_1 with a rational point P on it. Then (F,P), viewed as an elliptic curve, has no rational nontrivial 5-torsion points.

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Fact (Lemma 3.3.1):

Let $E/\mathbb{Q}$ be an elliptic curve defined over $\mathbb{Q}$, and $P \in E(\mathbb{Q})$ a 5-torsion point. Then there is a number $b \in \mathbb{Q}$ and an isomorphism $\varphi\colon E \to E'$, where E' is the curve defined by

$$E' : y^2 + (b+1)xy + by = x^3 + bx^2,$$

such that $\varphi(P) = (0,0)$.

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So if we assume that there is a nontrivial rational 5-torsion point on (F, P), it follows that there is a $b \in \mathbb{Q}^\times$ such that the j-invariant of (F, P) is

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This means that we have a rational point on the affine curve $C' \subset \mathcal{U} := \mathbb{A}^2(s, b) \cap \{sb \neq 0\}$ defined by

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The curve C' has no rational points.

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Using the map from C' to $\mathbb{A}^2(u, c)$

$$\theta \colon (s, b) \mapsto ((c_2^{-1}s^2 - c_2 s^{-2})^2, b - b^{-1}),$$

one can see that the existence of rational points on C' would imply the existence of rational points on the genus 0 curve $C \subset \mathbb{A}^2(u, c)$, where C is given by

$$C \colon 2(u + 96)^3(c + 11) = -(c^2 + 12c + 16)^3(u - 32)^2.$$

5-torsion points

Let $\overline{C} \subset \mathbb{P}^2(X, Y, Z)$ be the projective closure of the curve C, where $u = X/Z$ and $c = Y/Z$. Then there exists a birational morphism

$$\varphi\colon \mathbb{P}^1 \to \overline{C}, (p : q) \mapsto (X(p,q) : Y(p,q) : Z(p,q))$$

with

$$\begin{aligned}
X(p,q) =& 2^5 (p - 7168q)^2 (p^2 - 10240pq + 20971520q^2)^2 \cdot \\
& (p^2 - \frac{73728}{5} pq + 54525952q^2), \\
Y(p,q) =& -11(p - 8192q)^4 (p - \frac{36864}{5} q) \cdot \\
& (p^3 - 22528p^2 q + \frac{1862270976}{11} pq^2 - \frac{4668629450752}{11} q^3), \\
Z(p,q) =& (p - 8192q)^5 (p - 7168q)^2 (p - \frac{36864}{5} q).
\end{aligned}$$

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From the definition of the map θ and recalling the parametrization φ, the existence of a rational point on $\overline{C}$ coming from C' would imply the existence of a rational point on the curve $M \subset \mathbb{P}^2(p,q,r)$ defined by

$$M\colon q^2r^2 = 2\left(p^2 - \frac{73728}{5}pq + 54525952q^2\right)(p - 8192q)\left(p - \frac{36864}{5}q\right).$$

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Notice that on this curve there are at least two rational points, namely: $(1:0:8192)$ and $(5:0:36864)$, which correspond to the points $(8192:1), (36864:5) \in \mathbb{P}^1(p,q)$; both the points are sent to $(1:0:0) \in \overline{C}$ via φ.
We claim that these two are the only rational points on M.

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M is isomorphic over the rational to the elliptic curve

$$\boxed{G\colon y^2z = x^3 - 4x^2z + 20xz^2.}$$

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From the definition of the map θ and recalling the parametrization φ, the existence of a rational point on $\overline{C}$ coming from C' would imply the existence of a rational point on the curve $M \subset \mathbb{P}^2(p,q,r)$ defined by

$$M\colon q^2r^2 = 2(p^2 - \frac{73728}{5}pq + 54525952q^2)(p - 8192q)(p - \frac{36864}{5}q).$$

Notice that on this curve there are at least two rational points, namely: $(1:0:8192)$ and $(5:0:36864)$, which correspond to the points $(8192:1), (36864:5) \in \mathbb{P}^1(p,q)$; both the points are sent to $(1:0:0) \in \overline{C}$ via φ.
We claim that these two are the only rational points on M.
M is isomorphic over the rational to the elliptic curve

$$\boxed{G\colon y^2z = x^3 - 4x^2z + 20xz^2.}$$

But for the elliptic curve G one can see that

$$G(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}.$$

3-torsion points

Claim (Theorem 3.4.3):

Let F be a smooth fiber of ψ_1 with a rational point P on it. Then (F,P), viewed as an elliptic curve, has no rational nontrivial 3-torsion points.

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Fact (Lemma 3.4.1):

Let E be an elliptic curve defined over $\mathbb{Q}$ and P a rational 3-torsion point; assume E has nonzero j-invariant. Then there exist an element $b \in \mathbb{Q}^\times$ such that the pair (E, P) is isomorphic to the pair $(E_b, (0,0))$, where E_b is the elliptic curve defined by

$$y^2 + xy + \frac{b}{27}y = x^3.$$

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So if we assume that there is a nontrivial rational 3-torsion point on (F, P), it follows that there is a $b \in \mathbb{Q}^\times$ such that the j-invariant of (F, P) is

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This means that we have a rational point on the affine curve $C' \subset \mathcal{U} := \mathbb{A}^2(b, s) \cap \{bs \neq 0\}$ defined by

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Claim:

The curve C' has no rational points.

Using the map from C' to $\mathbb{A}^2$ defined by

$$\epsilon \colon (b, s) \mapsto (b, c_2^{-1}s^2 + c_2 s^{-2}),$$

one can see that the existence of rational points on C' would imply the existence of rational points on the genus 2 curve $C \subset \mathbb{A}^2(p,q)$, where C is given by

$$C \colon 2(u+96)^3(c+11) = -(c^2 + 12c + 16)^3(u-32)^2.$$

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- Using the Chabauty's method one can see that H admits only six rational points.
- These six points correspond on C to the points $(0, \pm 6), (1, \pm 6)$. This shows that these four points are the only rational points on C.
- These four points cannot come from rational points of C' via ϵ, since the equations $c_2^{-1}s^2 + c_2 s^{-2} = \pm 6$ have no rational solutions.

Full torsion subgroup

Mazur Theorem

Let $E/\mathbb{Q}$ be an elliptic curve defined over $\mathbb{Q}$. Then the torsion subgroup $E^{\mathrm{Tor}}(\mathbb{Q})$ of $E(\mathbb{Q})$ is isomorphic to one of the following fifteen groups:

$$\begin{array}{lll} \mathbb{Z}/N\mathbb{Z} & \text{with} & 1 \leq N \leq 10, N = 12 \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2N\mathbb{Z} & \text{with} & 1 \leq N \leq 4. \end{array}$$

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Theorem 3.5.2

Let $s \in \mathbb{Q}^{\times}$ and F be the fiber of ψ_1 over $(s:1)$ on W. Assume that F has at least one rational point, $P = (x_0 : y_0 : z_0 : w_0)$. Denote by $E = (F, P)$ the fiber viewed as elliptic curve. Then

$$E^{\mathrm{Tor}}(\mathbb{Q}) = \{(x_0 : y_0 : z_0 : w_0), (-x_0 : -y_0 : z_0 : w_0)\} \simeq \mathbb{Z}/2\mathbb{Z}.$$

Proof of the main theorem

Theorem 4.1.1

Let c_1, c_2 be two nonzero rationals and W be the surface defined as

$$W : x^4 - 4c_1^2 y^4 - c_2 z^4 - 4c_2 w^4 = 0.$$

Let $P = (x_0 : y_0 : z_0 : w_0)$ be a rational point on W with x_0, y_0 both nonzero.
If $|2c_1|$ is a square in $\mathbb{Q}^\times$, then also assume that z_0, w_0 are not both zero.
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So let F be a smooth fiber passing through P.
Consider the point $P' = (-x_0 : y_0 : z_0 : w_0) \in F$: from the hypotheses it follows that $P' \neq P, (-x_0 : -y_0 : z_0 : w_0)$, hence it has infinite order.

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So we have infinitely many rational points on F. To each but finitely many of them we can apply the same argument with respect to the other fibration. Then the Zariski density follows.

THE END

Thank you for the attention.