Potential density of rational points on K3 surfaces

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Rational points on $K3$ surfaces

We study the rational points of $K3$ surfaces defined over $\mathbb{Q}$.

Definition

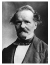
A $K3$ **surface** is a smooth projective surface X such that

- $H^1(X, \mathbb{Z}) = 0$,
- $K_X = 0$.

André Weil

Erich Kähler

Kunihiko Kodaira



Ernst Kummer

Mountain K2

Examples of K3 surfaces

Example 1

A smooth quartic surface in $\mathbb{P}^3$.
For example the Fermat quartic surface

$$X : x^4 + y^4 + z^4 + w^4 = 0$$

in $\mathbb{P}^3(x, y, z, w)$.

Example 2

Smooth complete intersection of type $(d_1, \dots, d_n)$ in $\mathbb{P}^{n+2}$ with $\sum d_i = n + 3$.

- $n = 1,\ d_1 = 4.$
- $n = 2,\ d_1 = 2,\ d_2 = 3.$
- $n = 3,\ d_1 = d_2 = d_3 = 2.$

Examples of K3 surfaces

Example 3

Kummer surfaces.
Let A be an abelian surface, with the involution $\iota\colon x \mapsto -x$ of A.
Let $\tilde{A} = A/\iota$.
The surface $\tilde{A}$ has 16 singular points: the images of the 2-torsion points of A.
Let $X \to \tilde{A}$ be the minimal resolution of $\tilde{A}$.
The surface X is a $K3$ surface.

Example 4

A double cover $X \to \mathbb{P}^2$ of $\mathbb{P}^2$ ramified along a smooth sextic.

First results

Notation/Definition

Let X be a $K3$ surface defined over $\mathbb{Q}$.
Denote by $X(\mathbb{Q})$ the set of its rational points.
We say that $X(\mathbb{Q})$ is **Zariski dense** if it is dense in the Zariski topology.

Theorem (F. - 2012)

Let $c_1, c_2 \in \mathbb{Q}^\times$ and let X be the surface of $\mathbb{P}^3(x, y, z, w)$ defined by

$$X : x^4 - c_1 y^4 - c_2 z^4 - 4c_2 w^4 = 0.$$

Let $P = (x_0 : y_0 : z_0 : w_0)$ be in $X(\mathbb{Q})$ and assume that $x_0, y_0 \in \mathbb{Q}^\times$. If $|c_1|$ is a square then also assume that $(z_0, w_0) \neq (0, 0)$. Then the set of rational points of X is Zariski dense.

First results

Theorem (Logan, McKinnon, van Luijk - 2010)

Let $a, b, c, d \in \mathbb{Q}^{\times}$ such that $abcd$ is a square, and let X be the surface of $\mathbb{P}^3(x, y, z, w)$ defined by

$$X : ax^4 + by^4 + cz^4 + dw^4 = 0.$$

Let $P = (x_0 : y_0 : z_0 : w_0)$ be in $X(\mathbb{Q})$ and assume that all its coordinates are nonzero. Also assume that P does not lie on any of the 48 lines of X. Then the set of rational points of X is dense both in the Zariski and the real analytic topology.

Open question

Question

Are there examples of $K3$ surfaces over $\mathbb{Q}$ with a nonempty set of rational points that is not Zariski dense?

Potential density

Notation/Definition

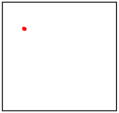
Let X be a $K3$ surface defined over $\mathbb{Q}$ and let K be a number field.
We denote with $X(K)$ the set of points of X defined over K.
We say that rational points of X are **potentially dense** if there is a number field K such that $X(K)$ is dense in the Zariski topology.

Theorem (Bogomolov, Tschinkel - 2000)

Let X be a $K3$ surface defined over a number field K. Assume that X admits an elliptic fibration or it has an infinite group of automorphisms. Then rational points of X are potentially dense.

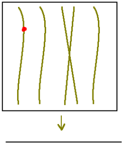
Idea of the proof

Elliptic fibrations



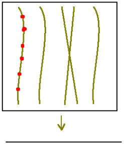
Idea of the proof

Elliptic fibrations



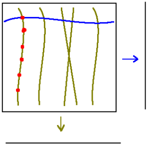
Idea of the proof

Elliptic fibrations



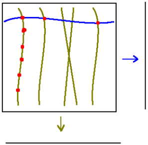
Idea of the proof

Elliptic fibrations



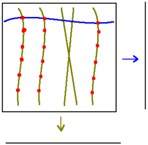
Idea of the proof

Elliptic fibrations



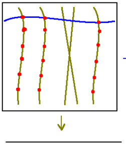
Idea of the proof

Elliptic fibrations

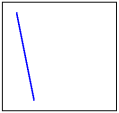


Idea of the proof

Elliptic fibrations

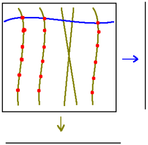


Automorphisms



Idea of the proof

Elliptic fibrations



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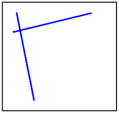
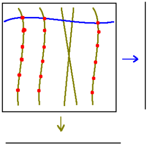


Idea of the proof

Elliptic fibrations



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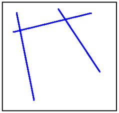
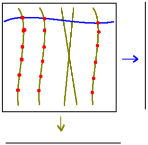


Idea of the proof

Elliptic fibrations



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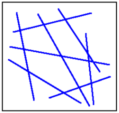
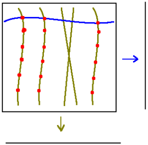


Idea of the proof

Elliptic fibrations



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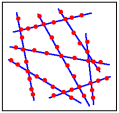

My project

The challenge

My project consists in understanding whether rational points are potentially dense or not in the cases not covered by the theorem of Bogomolov and Tschinkel.

Question

What are the cases not covered by the theorem?

We can classify the $K3$ surfaces not satisfying the hypotheses of the theorem by Bogomolov and Tschinkel by using their Néron-Severi lattices.

Notation/Definition

Let X be a $K3$ surface.
The **Néron-Severi** lattice of X, denoted by $\mathrm{NS}(X)$, is the Picard group of X endowed with the symmetric bilinear form coming from the intersection pairing of X.
Its rank, denoted by $\rho(X)$, is called the **Picard number of** X.

Remarks

- $\mathrm{NS}(X) \hookrightarrow H^2(X, \mathbb{Z}) \cong \Lambda_{K3}$;
- $1 \leq \rho(X) \leq 20$;
- The signature of $\mathrm{NS}(X)$ is $(1, \rho - 1)$.

My project

A $K3$ surface X does not satisfy the hypotheses of the theorem if and only if its Néron-Severi lattice is isomorphic to one of the following lattices:

- rank $=1$: $\left(\mathbb{Z},\left(2d\right)\right)$, with $d>0$.
- rank $=2$: $\left(\mathbb{Z}^2,\begin{pmatrix}-2 & a\\ a & b\end{pmatrix}\right)$, with $a\geq 0$ and $4b+a^2>0$.
- rank $=3$: $\left(\mathbb{Z}^3,\begin{pmatrix}6 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & -2\end{pmatrix}\right)$, $\left(\mathbb{Z}^3,\begin{pmatrix}36 & 0 & 0\\ 0 & -2 & 1\\ 0 & 1 & -2\end{pmatrix}\right)$, $\left(\mathbb{Z}^3,\begin{pmatrix}12 & 0 & 0\\ 0 & -2 & 1\\ 0 & 1 & -2\end{pmatrix}\right)$, $\left(\mathbb{Z}^3,\begin{pmatrix}4 & 0 & 0\\ 0 & -2 & 1\\ 0 & 1 & -2\end{pmatrix}\right)$, $\left(\mathbb{Z}^3,\begin{pmatrix}6 & 1 & -1\\ 1 & -2 & 1\\ -1 & 1 & -2\end{pmatrix}\right)$, $\left(\mathbb{Z}^3,\begin{pmatrix}14 & 1 & -1\\ 1 & -2 & 1\\ -1 & 1 & -2\end{pmatrix}\right)$.
- rank $=4$: $\left(\mathbb{Z}^4,\begin{pmatrix}2 & -1 & -1 & -1\\ -1 & -2 & 0 & 0\\ -1 & 0 & -2 & 0\\ -1 & 0 & 0 & -2\end{pmatrix}\right)$, $\left(\mathbb{Z}^4,\begin{pmatrix}12 & -2 & 0 & 0\\ -2 & -2 & -1 & 0\\ 0 & -1 & -2 & -1\\ 0 & 0 & -1 & -2\end{pmatrix}\right)$.

My project

The idea

Looking for family of elliptic curves parametrised by curves that are not birationally equivalent to $\mathbb{P}^1$.

Question

Knowing the Néron-Severi lattice of a K3 surface, what can we say about its rational points?

THE END

Thank you for your attention.