Solutions of Homework 1

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(1) Let X be the topological space given by the set $\{P, Q, \eta\}$ and the closed subset X, $\emptyset$, $\{P\}$, $\{Q\}$, $\{P, Q\}$. Let $\mathcal{F} = \underline{\mathbb{Z}}$ be the constant sheaf associate to $\mathbb{Z}$; let $\mathcal{G} = \mathbb{Z}^P \oplus \mathbb{Z}^Q$ be the sum of the skyscraper sheaves associated to $\mathbb{Z}$ at the points P and Q, respectively. Let $\phi\colon \mathcal{F} \to \mathcal{G}$ be the natural restriction. As the stalks of $\mathcal{G}$ at the points of X are $\mathcal{G}_P \cong \mathbb{Z}$, $\mathcal{G}_Q \cong \mathbb{Z}$, and $\mathcal{G}_\eta = 0$, the map φ is surjective. Notice that $\mathcal{F}(X) = \mathbb{Z}$ and $\mathcal{G}(X) = \mathbb{Z} \oplus \mathbb{Z}$, and so $\varphi(X)\colon \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ is the diagonal map, which is not surjective.

(2) (i) $\implies$ (ii). Suppose that $\operatorname{Spec} A$ is the disjoint union of two closed subsets $V(\mathfrak{a})$ and $V(\mathfrak{b})$. Applying (II.2.1) we see that $\mathfrak{ab} \subset \mathfrak{N}_A$, where $\mathfrak{N}_A$ is the nilradical of A, and $\mathfrak{a} + \mathfrak{b} = A$. (For an ideal $\mathfrak{a}$ it is elementary to see that $V(\mathfrak{a}) = \operatorname{Spec} A$ if and only if $\mathfrak{a} \subset \mathfrak{N}_A$ and $V(\mathfrak{a}) = \emptyset$ if and only if $\mathfrak{a} = A$). Hence, we can find $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$ such that $a + b = 1$ and ab is a nilpotent element, say $(ab)^n = 0$. We let $e_1 := \sum_{i=0}^{n-1} \binom{N}{i} a^{N-i} b^i$ and $e_2 := \sum_{i=0}^{n-1} \binom{N}{i} a^i b^{N-i}$; then for N big enough ($N \geq 2n - 1$) we have that $1 = 1^N = (a + b)^N = e_1 + e_2$ and $e_1 e_2 = 0$ because all terms where a factor $a^n b^n$ appears are equal to zero. It follows that e_1 and e_2 are orthogonal idempotents, as $e_1^2 = e_1(1 - e_2) = e_1$ and analogously for e_2.
(ii) $\implies$ (i). A prime ideal $\mathfrak{p}$ of A contains $0 = e_1 e_2$, so $\mathfrak{p}$ must contain either e_1 or e_2. On the other hand, $\mathfrak{p}$ cannot contain both, as $e_1 + e_2 = 1 \notin \mathfrak{p}$. It follows that $\operatorname{Spec} A$ is the disjoint union of the two closed subsets $V((e_1))$ and $V((e_2))$, hence it is disconnected.
(iii) $\implies$ (ii). If $A \cong A_1 \times A_2$, then $(1, 0)$ and $(0, 1)$ are a pair of orthogonal idempotents.
(ii) $\implies$ (iii). We let $A_1 := \{f \in A | f e_2 = 0\}$ and $A_2 := \{f \in A | f e_1 = 0\}$. Note that $e_1 \in A_1$ and $f e_1 = f(1 - e_2) = f$ for every $f \in A_1$, which shows that A_1 is a ring whose unit is e_1. Analogously A_2 is a ring whose unit is e_2. We have two maps $A \to A_1 \times A_2$ defined by $f \mapsto (f e_1, f e_2)$ and $A_1 \times A_2 \to A$ defined by $(f, g) \mapsto f + g$, which are easily seen to be ring homomorphisms inverse to each other.

(3) First we claim that a prime ideal $\mathfrak{p}$ of $\mathbb{Z}[X]$ has one of the following forms:

1. $\mathfrak{p} = 0$;
2. $\mathfrak{p} = (p)$, for some prime integer p;
3. $\mathfrak{p} = (f)$, for some irreducible polynomial $f \in \mathbb{Z}[X]$.
4. $\mathfrak{p} = (p, f)$, where p is a prime integer and f is an irreducible polynomial of $\mathbb{Z}[X]$ such that its reduction in $\mathbb{F}_p[X]$ is also irreducible.

In order to see this, consider the inclusion $\iota\colon \mathbb{Z} \hookrightarrow \mathbb{Z}[X]$, and let $\mathfrak{p}$ be a prime ideal of $\mathbb{Z}[X]$. Then $\iota^{-1}(\mathfrak{p})$ is a prime ideal of $\mathbb{Z}$ and hence, as $\mathbb{Z}$ is a PID, $\iota^{-1}(\mathfrak{p})$ is either (0) or (p) for some integer prime p. Assume $\iota^{-1}(\mathfrak{p}) = (p)$. This implies that $\mathfrak{p}$ contains $p\mathbb{Z}[X]$. Such ideals are in a 1-to-1 correspondence to prime ideals of $\mathbb{F}_p[X]$. As $\mathbb{F}_p[X]$ is a PID, its prime ideals are the zero ideal and the ideals that are generated by a single irreducible polynomial. If $\mathfrak{p}$ corresponds to the zero ideal in $\mathbb{F}_p[X]$, then $\mathfrak{p} = (p)$; if $\mathfrak{p}$ corresponds to the ideal $(g) \subset \mathbb{F}_p[X]$, then $\mathfrak{p} = (p, f)$ where $f \in \mathbb{Z}[X]$ is such that $\bar{f} = g$. Assume now that $\iota^{-1}(\mathfrak{p}) = (0)$. This means that $\mathfrak{p} \cap \mathbb{Z} = \{0\}$. Therefore $\mathfrak{p}$ corresponds to a prime ideal $\mathbb{Q}[X]$, that is a PID. Hence, $\mathfrak{p}$ is the ideal (f), where f is an irreducible ideal of $\mathbb{Q}[X]$ (or, equivalently, of $\mathbb{Z}[X]$).

We study now the topology of $\operatorname{Spec} \mathbb{Z}[X]$ and its structure sheaf. The zero ideal (0) is contained in every ideal of $\mathbb{Z}[X]$ and therefore it is dense in $\operatorname{Spec} \mathbb{Z}[X]$. The stalk $\mathcal{O}_{(0)}$ is isomorphic to

$\mathbb{Q}(X)$. The ideals of the form (p, f) are maximal, and therefore they are closed points of $\operatorname{Spec}\mathbb{Z}[X]$. The residue field $k((p, f))$ is isomorphic to $\mathbb{F}_{p^d}$, where d is the degree of f. If $\mathfrak{p} = (p)$, then $\mathfrak{p}$ is contained in all the ideals of the form (p, f). The residue field $k((p))$ is isomorphic to $\mathbb{F}_p[X]$. If $\mathfrak{p} = (f)$. then $\mathfrak{p}$ is contained in all the ideals of the form (p, f), where p is a prime integer such that the reduction $f \bmod p$ in $\mathbb{F}_p[X]$ is irreducible. The residue field $k((f))$ is isomorphic to $\mathbb{Q}(\alpha)$, where α is a root of f.

(4) Assume every element of S_+ is nilpotent. Then S_+ is contained in the nilradical of S and hence, in turn, in every homogeneous prime ideal of S. Conversely saying that $\operatorname{Proj} S = \emptyset$ is equivalent to saying that every prime homogeneous ideal of S contains S_+. Therefore S_+ is contained in the nilradical of S and hence every element of S_+ is nilpotent.