Unirationality of del Pezzo surfaces of degree 2 over finite fields

Dino Festi
joint with Ronald van Luijk

ALGANT

Universiteit Leiden - Università di Milano

February 22, 2014

Unirationality

Let X be a (projective, smooth, geometrically irreducible) variety over a field k, and let $\bar{k}$ be an algebraic closure of k.

Unirationality

Let X be a (projective, smooth, geometrically irreducible) variety over a field k, and let $\bar{k}$ be an algebraic closure of k.

Definitions

The variety X is said to be *rational* if it is birational over k to the projective space $\mathbb{P}^m$ for some $m \geq 0$.

Unirationality

Let X be a (projective, smooth, geometrically irreducible) variety over a field k, and let $\bar{k}$ be an algebraic closure of k.

Definitions

The variety X is said to be *rational* if it is birational over k to the projective space $\mathbb{P}^m$ for some $m \geq 0$.

The variety X is said to be *geometrically rational* if $X \times_k \bar{k}$ is birational to $\mathbb{P}^m_{\bar{k}}$ for some $m \geq 0$.

Unirationality

Let X be a (projective, smooth, geometrically irreducible) variety over a field k, and let $\bar{k}$ be an algebraic closure of k.

Definitions

The variety X is said to be *rational* if it is birational over k to the projective space $\mathbb{P}^m$ for some $m \geq 0$.

The variety X is said to be *geometrically rational* if $X \times_k \bar{k}$ is birational to $\mathbb{P}^m_{\bar{k}}$ for some $m \geq 0$.

The variety X is said to be *unirational* if there is a dominant map $\mathbb{P}^m \dashrightarrow X$ for some $m \geq 0$.

Del Pezzo surfaces

Definition

A del Pezzo surface over k is a variety X over k, of dimension 2, with ample anticanonical divisor $-K_X$.

Del Pezzo surfaces

Definition

A del Pezzo surface over k is a variety X over k, of dimension 2, with ample anticanonical divisor $-K_X$.

Question

Why are the del Pezzo surfaces important?

Del Pezzo surfaces

Definition

A del Pezzo surface over k is a variety X over k, of dimension 2, with ample anticanonical divisor $-K_X$.

Question

Why are the del Pezzo surfaces important?

Theorem (Iskovskikh)

Let X be a smooth projective geometrically rational surface over a field k. Then X is k–birational to either a del Pezzo surface or a rational conic bundle.

Del Pezzo surfaces

Definition

Let X be a del Pezzo surface with canonical divisor K_X.
The degree of X is $\deg X = K_X^2 = d$.

Del Pezzo surfaces

Definition

Let X be a del Pezzo surface with canonical divisor K_X.
The degree of X is $\deg X = K_X^2 = d$.

Del Pezzo surfaces are classified by their degree.

Del Pezzo surfaces

Definition

Let X be a del Pezzo surface with canonical divisor K_X.
The degree of X is $\deg X = K_X^2 = d$.

Del Pezzo surfaces are classified by their degree.
Let $P_1, \ldots, P_r$, with $r \leq 8$, be points in $\mathbb{P}^2$. They are said to be in general position if

- No three are on a line;

Del Pezzo surfaces

Definition

Let X be a del Pezzo surface with canonical divisor K_X.
The degree of X is $\deg X = K_X^2 = d$.

Del Pezzo surfaces are classified by their degree.
Let $P_1, \ldots, P_r$, with $r \leq 8$, be points in $\mathbb{P}^2$. They are said to be in general position if

- No three are on a line;
- No six are on a quadric;

Del Pezzo surfaces

Definition

Let X be a del Pezzo surface with canonical divisor K_X.
The degree of X is $\deg X = K_X^2 = d$.

Del Pezzo surfaces are classified by their degree.
Let $P_1, \ldots, P_r$, with $r \leq 8$, be points in $\mathbb{P}^2$. They are said to be in general position if

- No three are on a line;
- No six are on a quadric;
- No eight are on a singular cubic with the singularity at one of these points.

Del Pezzo surfaces

Definition

Let X be a del Pezzo surface with canonical divisor K_X.
The degree of X is $\deg X = K_X^2 = d$.

Del Pezzo surfaces are classified by their degree.
Let $P_1, \ldots, P_r$, with $r \leq 8$, be points in $\mathbb{P}^2$. They are said to be in general position if

- No three are on a line;
- No six are on a quadric;
- No eight are on a singular cubic with the singularity at one of these points.

Del Pezzo surfaces

Definition

Let X be a del Pezzo surface with canonical divisor K_X.
The degree of X is $\deg X = K_X^2 = d$.

Del Pezzo surfaces are classified by their degree.
Let $P_1, \ldots, P_r$, with $r \leq 8$, be points in $\mathbb{P}^2$. They are said to be in general position if

- No three are on a line;
- No six are on a quadric;
- No eight are on a singular cubic with the singularity at one of these points.

Theorem

Let X be a del Pezzo surface over k, and let k^{sep} be a separable closure of k.
Then X is isomorphic over k^{sep} to either

$$\mathbb{P}^1 \times \mathbb{P}^1, \text{ in which case } d = 8;$$

or to

$$\mathbb{P}^2 \text{ blown up at } r \leq 8 \text{ points in general position, in which case } d = 9 - r.$$

Unirationality of del Pezzo surfaces

We are interested in studying the rationality of del Pezzo surfaces.

We have a partial converse of the Iskovskikh's theorem.

Unirationality of del Pezzo surfaces

We are interested in studying the rationality of del Pezzo surfaces.

We have a partial converse of the Iskovskikh's theorem.

Theorem

Every del Pezzo surface is geometrically rational.

Unirationality of del Pezzo surfaces

We are interested in studying the rationality of del Pezzo surfaces.

We have a partial converse of the Iskovskikh's theorem.

Theorem

Every del Pezzo surface is geometrically rational.

What about over the ground field?

Unirationality of del Pezzo surfaces

We are interested in studying the rationality of del Pezzo surfaces.

We have a partial converse of the Iskovskikh's theorem.

Theorem

Every del Pezzo surface is geometrically rational.

What about over the ground field?

Theorem (Kollár, Manin, Segre)

Let X be a del Pezzo surface of degree $d \geq 3$ defined over a field k, and assume that its set of k–rational points $X(k)$ is non–empty.
Then X is unirational over k.

Unirationality of del Pezzo surfaces of degree 2

The previous theorem gives a complete answer for the unirationality of del Pezzo surfaces of degree $d \geq 3$.
The situation for del Pezzo surfaces of degree 2 is a bit more complicated.

Unirationality of del Pezzo surfaces of degree 2

The previous theorem gives a complete answer for the unirationality of del Pezzo surfaces of degree $d \geq 3$.
The situation for del Pezzo surfaces of degree 2 is a bit more complicated.

A projective model for del Pezzo surfaces of degree 2

- Using Riemann–Roch for surfaces, one can show that X is isomorphic to a surface in $\mathbb{P}(1,1,1,2)$ with coordinates x, y, z, w of the form

  $$w^2 = F(x, y, z),$$

  where F is a homogeneous degree 4 polynomial in $k[x, y, z]$.

Unirationality of del Pezzo surfaces of degree 2

The previous theorem gives a complete answer for the unirationality of del Pezzo surfaces of degree $d \geq 3$.
The situation for del Pezzo surfaces of degree 2 is a bit more complicated.

A projective model for del Pezzo surfaces of degree 2

- Using Riemann–Roch for surfaces, one can show that X is isomorphic to a surface in $\mathbb{P}(1,1,1,2)$ with coordinates x, y, z, w of the form
$$w^2 = F(x,y,z),$$
where F is a homogeneous degree 4 polynomial in $k[x,y,z]$.
- We identify X with its image in $\mathbb{P}(1,1,1,2)$.
The map $\pi : X \to \mathbb{P}^2$ given by
$$(x:y:z:w) \mapsto (x:y:z)$$
is a double covering of $\mathbb{P}^2$, ramified along the *branch curve* $B : F = 0$ in $\mathbb{P}^2$.

Unirationality of del Pezzo surfaces of degree 2

The previous theorem gives a complete answer for the unirationality of del Pezzo surfaces of degree $d \geq 3$.
The situation for del Pezzo surfaces of degree 2 is a bit more complicated.

A projective model for del Pezzo surfaces of degree 2

- Using Riemann–Roch for surfaces, one can show that X is isomorphic to a surface in $\mathbb{P}(1,1,1,2)$ with coordinates x, y, z, w of the form
$$w^2 = F(x,y,z),$$
where F is a homogeneous degree 4 polynomial in $k[x,y,z]$.
- We identify X with its image in $\mathbb{P}(1,1,1,2)$.
The map $\pi : X \to \mathbb{P}^2$ given by
$$(x:y:z:w) \mapsto (x:y:z)$$
is a double covering of $\mathbb{P}^2$, ramified along the *branch curve* $B : F = 0$ in $\mathbb{P}^2$.
- The preimage $R := \pi^{-1}(B)$ of the curve B in X is called the *ramification curve* of π.

Unirationality of del Pezzo surfaces of degree 2

Theorem (Manin, 1986)

Let X be a del Pezzo surface of degree 2 defined over a field k, and assume that there exists a rational point $P \in X(k)$ not on any exceptional line and not on the ramification curve R.
Then X is unirational over k.

Unirationality of del Pezzo surfaces of degree 2

Theorem (Manin, 1986)

Let X be a del Pezzo surface of degree 2 defined over a field k, and assume that there exists a rational point $P \in X(k)$ not on any exceptional line and not on the ramification curve R.
Then X is unirational over k.

Remarks

- *Bad points*: the ramification curve R and all (56) exceptional lines of X.

Unirationality of del Pezzo surfaces of degree 2

Theorem (Manin, 1986)

Let X be a del Pezzo surface of degree 2 defined over a field k, and assume that there exists a rational point $P \in X(k)$ not on any exceptional line and not on the ramification curve R.
Then X is unirational over k.

Remarks

- *Bad points*: the ramification curve R and all (56) exceptional lines of X.
- In explaining the proof of the theorem we will only consider the case when char $k \neq 2$ and k is infinite.

Unirationality of del Pezzo surfaces of degree 2

Idea of the proof

- Consider ϕ: $X' \to X$, the blow up of X at P.
  From the hypotheses on P it follows that X' is a del Pezzo surface of degree 1.

Idea of the proof

- Consider $\phi\colon X' \to X$, the blow up of X at P.
From the hypotheses on P it follows that X' is a del Pezzo surface of degree 1.
- Using again Riemann–Roch for surfaces one can show that every del Pezzo surface of degree 1, and in particular X', is isomorphic to a surface in $\mathbb{P}(2,3,1,1)$ with coordinates x', y', z', w' of the form

$$y'^2 = x'^3 + f(z',w')x'^2 + g(z',w')x' + h(z',w'),$$

where f, g, h are homogenous polynomials in $k[z',w']$ of degree 2, 4 and 6 respectively.

Idea of the proof

- Consider $\phi\colon X' \to X$, the blow up of X at P.
  From the hypotheses on P it follows that X' is a del Pezzo surface of degree 1.
- Using again Riemann–Roch for surfaces one can show that every del Pezzo surface of degree 1, and in particular X', is isomorphic to a surface in $\mathbb{P}(2,3,1,1)$ with coordinates x', y', z', w' of the form

$$y'^2 = x'^3 + f(z',w')x'^2 + g(z',w')x' + h(z',w'),$$

  where f, g, h are homogenous polynomials in $k[z',w']$ of degree 2, 4 and 6 respectively.
- If we identify X' with its image in $\mathbb{P}(2,3,1,1)$ then the involution $i\colon (x' : y' : z' : w') \mapsto (x' : -y' : z' : w')$ of $\mathbb{P}(2,3,1,1)$ acts on X'.

Idea of the proof

- Consider $\phi\colon X' \to X$, the blow up of X at P.
  From the hypotheses on P it follows that X' is a del Pezzo surface of degree 1.
- Using again Riemann–Roch for surfaces one can show that every del Pezzo surface of degree 1, and in particular X', is isomorphic to a surface in $\mathbb{P}(2,3,1,1)$ with coordinates x', y', z', w' of the form

$$y'^2 = x'^3 + f(z', w')x'^2 + g(z', w')x' + h(z', w'),$$

  where f, g, h are homogenous polynomials in $k[z', w']$ of degree 2, 4 and 6 respectively.
- If we identify X' with its image in $\mathbb{P}(2,3,1,1)$ then the involution $i\colon (x' : y' : z' : w') \mapsto (x' : -y' : z' : w')$ of $\mathbb{P}(2,3,1,1)$ acts on X'.
- Let E be the exceptional divisor of X' lying over P, and consider the curve $D' = i(E) \subset X'$.
  We have that $g(D') = g(E) = 0$ and that $D' = i(E) \neq E$.

- The curve $D = \phi(D')$ is rational, hence it is birational to $\mathbb{P}^1$ and we can parametrize it.

Unirationality of del Pezzo surfaces of degree 2

Idea of the proof

- The curve $D = \phi(D')$ is rational, hence it is birational to $\mathbb{P}^1$ and we can parametrize it.
- In this way we get a map $\mathbb{P}^1 \to X$ birational onto its image D.

- The curve $D = \phi(D')$ is rational, hence it is birational to $\mathbb{P}^1$ and we can parametrize it.
- In this way we get a map $\mathbb{P}^1 \to X$ birational onto its image D.
- Applying the same argument used for P to every point of D we get the desired map $\mathbb{P}^2 \dashrightarrow X$.

Unirationality of del Pezzo surfaces of degree 2

Theorem (Salgado, Testa, Várilly–Alvarado, 2013)

Let X be a del Pezzo surface of degree 2 defined over a field k and let P be a rational point on X. If the point P is not contained in four exceptional lines nor on the ramification curve R then X is unirational over k.

Unirationality of del Pezzo surfaces of degree 2

Theorem (Salgado, Testa, Várilly–Alvarado, 2013)

Let X be a del Pezzo surface of degree 2 defined over a field k and let P be a rational point on X. If the point P is not contained in four exceptional lines nor on the ramification curve R then X is unirational over k.

Remarks

Bad points: generalized Eckardt points and the ramification curve R.

Notice that this locus is much smaller than the locus given by Manin's theorem.

Unirationality of del Pezzo surfaces of degree 2

By the same authors we have also the following result.

By the same authors we have also the following result.

Lemma

Let X be a del Pezzo surface of degree 2 defined over a field k and suppose that $X(k)$ contains eight rational points $P_1, \dots, P_8$ with the property that $\pi(P_1), \dots, \pi(P_8)$ are distinct points not lying on the branch curve B. Then one of the points $P_1, \dots, P_8$ is either contained in an exceptional line defined over k, or it is not a generalized Eckardt point.
In particular, the surface X is unirational.

Unirationality of del Pezzo surfaces of degree 2 over finite fields

Theorem (Salgado, Testa, Várilly–Alvarado)

Let X be a del Pezzo surface of degree 2 defined over a finite field $\mathbb{F}$.
The surface X is unirational except possibly in the following cases up to isomorphisms:

$$\begin{aligned} X_1/\mathbb{F}_3 \colon -w^2 &= (x^2+y^2)^2 + y^3 z - y z^3, \\ X_2/\mathbb{F}_3 \colon -w^2 &= x^4 + y^3 z - y z^3, \\ X_3/\mathbb{F}_9 \colon c w^2 &= x^4 + y^4 + z^4, \end{aligned}$$

where $c \in \mathbb{F}_9$ is a non–square.

Remarks

This theorem gives almost a complete answer to the question about unirationality of del Pezzo surfaces of degree 2 over finite fields.
After this theorem the unirationality of only three surfaces (up to isomorphisms) is still uncertain.

Remarks

This theorem gives almost a complete answer to the question about unirationality of del Pezzo surfaces of degree 2 over finite fields.
After this theorem the unirationality of only three surfaces (up to isomorphisms) is still uncertain.
These three surfaces have very few rational points, namely:

- $\#X_1(\mathbb{F}_3) = 1$,

Remarks

This theorem gives almost a complete answer to the question about unirationality of del Pezzo surfaces of degree 2 over finite fields.
After this theorem the unirationality of only three surfaces (up to isomorphisms) is still uncertain.
These three surfaces have very few rational points, namely:

- $\#X_1(\mathbb{F}_3) = 1$,
- $\#X_2(\mathbb{F}_3) = 4$,

Remarks

This theorem gives almost a complete answer to the question about unirationality of del Pezzo surfaces of degree 2 over finite fields.
After this theorem the unirationality of only three surfaces (up to isomorphisms) is still uncertain.
These three surfaces have very few rational points, namely:

- $\#X_1(\mathbb{F}_3) = 1$,
- $\#X_2(\mathbb{F}_3) = 4$,
- $\#X_3(\mathbb{F}_9) = 28$;

Remarks

This theorem gives almost a complete answer to the question about unirationality of del Pezzo surfaces of degree 2 over finite fields.
After this theorem the unirationality of only three surfaces (up to isomorphisms) is still uncertain.
These three surfaces have very few rational points, namely:

- $\#X_1(\mathbb{F}_3) = 1$,
- $\#X_2(\mathbb{F}_3) = 4$,
- $\#X_3(\mathbb{F}_9) = 28$;

Remarks

This theorem gives almost a complete answer to the question about unirationality of del Pezzo surfaces of degree 2 over finite fields.
After this theorem the unirationality of only three surfaces (up to isomorphisms) is still uncertain.
These three surfaces have very few rational points, namely:

- $\#X_1(\mathbb{F}_3) = 1$,
- $\#X_2(\mathbb{F}_3) = 4$,
- $\#X_3(\mathbb{F}_9) = 28$;

and they are all on the ramification curve!

Remarks

This theorem gives almost a complete answer to the question about unirationality of del Pezzo surfaces of degree 2 over finite fields.
After this theorem the unirationality of only three surfaces (up to isomorphisms) is still uncertain.
These three surfaces have very few rational points, namely:

- $\#X_1(\mathbb{F}_3) = 1$,
- $\#X_2(\mathbb{F}_3) = 4$,
- $\#X_3(\mathbb{F}_9) = 28$;

and they are all on the ramification curve!

Question

Can we get some more information about these three surfaces?

Unirationality of del Pezzo surfaces of degree 2 over finite fields

Answer

YES!

Unirationality of del Pezzo surfaces of degree 2 over finite fields

Answer

YES!

By the work of Manin and Salgado, Testa, Várilly–Alvarado, the problem of proving the unirationality of these surfaces reduces to the problem of finding a non-exceptional rational curve on the surface.

Answer

YES!

By the work of Manin and Salgado, Testa, Várilly–Alvarado, the problem of proving the unirationality of these surfaces reduces to the problem of finding a non-exceptional rational curve on the surface.

Using the projection π we can look at curves in $\mathbb{P}^2$ in order to find curves on X.

We look for a rational curve D in $\mathbb{P}^2(x, y, z)$ such that:

We look for a rational curve D in $\mathbb{P}^2(x, y, z)$ such that:

1. D has degree 4;

We look for a rational curve D in $\mathbb{P}^2(x, y, z)$ such that:

1. D has degree 4;
2. D has an ordinary triple point at a rational point P of B, the branch locus of X with respect to π;

We look for a rational curve D in $\mathbb{P}^2(x, y, z)$ such that:

1. D has degree 4;
2. D has an ordinary triple point at a rational point P of B, the branch locus of X with respect to π;
3. the tangent line to B at P is tangent to one of the three branches of D at P;

We look for a rational curve D in $\mathbb{P}^2(x, y, z)$ such that:

1. D has degree 4;
2. D has an ordinary triple point at a rational point P of B, the branch locus of X with respect to π;
3. the tangent line to B at P is tangent to one of the three branches of D at P;
4. the intersection multiplicities at the intersection points of D and B are all even;

We look for a rational curve D in $\mathbb{P}^2(x,y,z)$ such that:

1. D has degree 4;
2. D has an ordinary triple point at a rational point P of B, the branch locus of X with respect to π;
3. the tangent line to B at P is tangent to one of the three branches of D at P;
4. the intersection multiplicities at the intersection points of D and B are all even;
5. the preimage $C = \pi^{-1}(D) \subset X$ of D is an irreducible curve on X.

We look for a rational curve D in $\mathbb{P}^2(x,y,z)$ such that:

1. D has degree 4;
2. D has an ordinary triple point at a rational point P of B, the branch locus of X with respect to π;
3. the tangent line to B at P is tangent to one of the three branches of D at P;
4. the intersection multiplicities at the intersection points of D and B are all even;
5. the preimage $C = \pi^{-1}(D) \subset X$ of D is an irreducible curve on X.

We look for a rational curve D in $\mathbb{P}^2(x, y, z)$ such that:

1. D has degree 4;
2. D has an ordinary triple point at a rational point P of B, the branch locus of X with respect to π;
3. the tangent line to B at P is tangent to one of the three branches of D at P;
4. the intersection multiplicities at the intersection points of D and B are all even;
5. the preimage $C = \pi^{-1}(D) \subset X$ of D is an irreducible curve on X.

We can assume that $P = (0:0:1)$ and let $l(x, y, z) = 0$ be the tangent line to B at P. Then the first three conditions imply that D is defined by an equation of the form:

We look for a rational curve D in $\mathbb{P}^2(x, y, z)$ such that:

1. D has degree 4;
2. D has an ordinary triple point at a rational point P of B, the branch locus of X with respect to π;
3. the tangent line to B at P is tangent to one of the three branches of D at P;
4. the intersection multiplicities at the intersection points of D and B are all even;
5. the preimage $C = \pi^{-1}(D) \subset X$ of D is an irreducible curve on X.

We can assume that $P = (0:0:1)$ and let $l(x, y, z) = 0$ be the tangent line to B at P. Then the first three conditions imply that D is defined by an equation of the form:

$$f := x^4 + l \cdot (f_3 + z \cdot f_2) = 0,$$

with f_i homogeneous polynomial in x and y of degree i.

Running through all the possible coefficients of the polynomials f_2 and f_3 (seven coefficients) we got the desired curves for all the three cases.

Running through all the possible coefficients of the polynomials f_2 and f_3 (seven coefficients) we got the desired curves for all the three cases.

Theorem (F., van Luijk)

The afore defined surfaces $X_1/\mathbb{F}_3$, $X_2/\mathbb{F}_3$ and $X_3/\mathbb{F}_9$ are unirational over their field of definition.

Unirationality of del Pezzo surfaces of degree 2 over finite fields

Putting together all the results we get the following compact corollary:

Putting together all the results we get the following compact corollary:

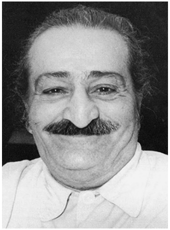
Corollary (F., van Luijk)

Every del Pezzo surface of degree 2 over a finite field $\mathbb{F}$ is unirational over $\mathbb{F}$.

Putting together all the results we get the following compact corollary:

Corollary (F., van Luijk)

Every del Pezzo surface of degree 2 over a finite field $\mathbb{F}$ is unirational over $\mathbb{F}$.



THE END

Thank you for your attention.