Exercises on § III.6

1. Prove the following statements.

   (i) For fixed $\mathcal{F} \in \mathfrak{Mod}(X)$, the functor $\operatorname{Hom}(\mathcal{F}, \cdot)$ is a left exact covariant functor from $\mathfrak{Mod}(X)$ to $\mathfrak{Ab}$.

   (ii) For fixed $\mathcal{F} \in \mathfrak{Mod}(X)$, the functor $\mathcal{H}om(\mathcal{F}, \cdot)$ is a left exact covariant functor from $\mathfrak{Mod}(X)$ to $\mathfrak{Mod}(X)$.

   (iii) The functors $\operatorname{Hom}(\mathcal{O}_X, \cdot)$ and $\Gamma(X, \cdot)$ from $\mathfrak{Mod}(X)$ to $\mathfrak{Ab}$ are equal.

   (iv) The functor $\mathcal{H}om(\mathcal{O}_X, \cdot)$ is the identity functor of $\mathfrak{Mod}(X)$.

2. (Addendum to Proposition III.6.9) Let k be a field and $X = \mathbb{P}^1_k$. Find two coherent sheaves $\mathcal{F}, \mathcal{G}$ on X such that $\operatorname{Ext}^i(\mathcal{F}, \mathcal{G})$ is *not* isomorphic to $\Gamma(X, \mathcal{E}xt^i(\mathcal{F}, \mathcal{G}))$.

3. (Exercise III.6.4) Let X be a noetherian scheme, and suppose that every coherent sheaf on X is a quotient of a locally free sheaf *of finite rank*.[1] In this case we say $\mathfrak{Coh}(X)$ has enough locally frees. then for any $\mathcal{G} \in \mathfrak{Mod}(X)$, show that the δ-functor $(\mathcal{E}xt^i(\cdot, \mathcal{G}))$, from $\mathfrak{Coh}(X)$ to $\mathfrak{Mod}(X)$, is a contravariant universal δ-functor. [Hint: Show $\mathcal{E}xt^i(\cdot, \mathcal{G})$ is coeffaceable for $i > 0$.]

4. (Exercise III.6.5) Let X be a noetherian scheme, and assume that $\mathfrak{Coh}(X)$ has enough locally frees (Ex. 6.4). Then for any coherent sheaf $\mathcal{F}$ we define the homological dimension of $\mathcal{F}$, denoted $\operatorname{hd}(\mathcal{F})$, to be the least length of a locally free resolution of $\mathcal{F}$ (or $+\infty$ if there is no finite one). Show:

   (a) $\mathcal{F}$ is locally free if and only if $\mathcal{E}xt^1(\mathcal{F}, \mathcal{G}) = 0$ for all $\mathcal{G} \in \mathfrak{Mod}(X)$; [Hint: Use Proposition 6.10A and the fact that a (finitely generated) projective module over a local ring is free.]

   (b) $\operatorname{hd}(\mathcal{F}) \leq n$ if and only if $\mathcal{E}xt^i(\mathcal{F}, \mathcal{G}) = 0$ for all $i > n$ and all $\mathcal{G} \in \mathfrak{Mod}(X)$; [Hint: Use (a) and induction on n.]

   (c) $\operatorname{hd}(\mathcal{F}) = \sup_x \operatorname{pd}_{\mathcal{O}_x} \mathcal{F}_x$. [Hint: Use (b).]

[1] This is missing in Hartshorne. Or is it not necessary?