Reading guidelines 2

June 5, 2017

Surjectivity and morphisms of affine schemes

Let $\varphi\colon A \to B$ be a ring homomorphism and $(f, f^{\#})\colon (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ its associated morphism of schemes, with $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$. We have seen that φ is injective if and only if $f^{\#}$ is injective. Prove the following statements.

(i) ($\star$) If φ is surjective, then $f^{\#}$ is surjective. (Hint: look at the stalks $\mathcal{O}_{Y,\mathfrak{q}} \to (f_*\mathcal{O}_X)_{\mathfrak{q}}$ for $\mathfrak{q} \in Y$, and use the computations that we have seen together.)

(ii) ($\star\star$) If $f^{\#}$ is surjective, then φ is surjective. (Hint: consider the factorization $Y \xrightarrow{g} Y' \xrightarrow{h} X$ of f with $Y' = \operatorname{Spec}(A/\ker\varphi)$. Prove that ($\star$) the morphism $g^{\#}$ is injective and that ($\star\star$) $h_*g^{\#}\colon h_*\mathcal{O}_{X'} \to h_*g_*\mathcal{O}_X = f_*\mathcal{O}_Y$ is both surjective and injective, i.e. it is an isomorphism of sheaves.)

(iii) ($\star$) If φ is surjective, then f is a homeomorphism of Y onto a closed subset of X, i.e $(f, f^{\#})$ is a closed immersion.

($\star\star\star$) Find an example of a morphism of schemes $(f, f^{\#})\colon (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ which is *not* a closed immersion and such that $f^{\#}$ is surjective.

Morphisms of affine schemes that you will meet in real life and the fibre of a morphism

Prove the following statements.

(i) ($\star$) If S is a multiplicatively closed subset of a ring A, then the canonical homomorphism $A \to S^{-1}A$ induces a homeomorphism of $\operatorname{Spec} S^{-1}A$ onto its image in $\operatorname{Spec} A$.

(ii) ($\star$) Let $\varphi\colon A \to B$ be a ring homomorphism and $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ its induced morphism of schemes. Consider $\operatorname{Spec} S^{-1}A$ and $\operatorname{Spec} S^{-1}B$ as subsets of $\operatorname{Spec} A$ and $\operatorname{Spec} B$, respectively (by $S^{-1}B$ we actually mean $\varphi(S)^{-1}B$). Prove that $f^{-1}(\operatorname{Spec} S^{-1}A) = \operatorname{Spec} S^{-1}B$ and that the restriction of f to $\operatorname{Spec} S^{-1}B$ is induced by the homomorphism $S^{-1}\varphi\colon S^{-1}A \to S^{-1}B$.

(iii) ($\star$) Let $\mathfrak{a}$ be an ideal of A and $\mathfrak{b}$ its extension in B. Identify $\operatorname{Spec} A/\mathfrak{a}$ with $V(\mathfrak{a}) \subset \operatorname{Spec} A$ and $\operatorname{Spec} B/\mathfrak{b}$ with $V(\mathfrak{b}) \subset \operatorname{Spec} B$. Then the restriction of f to $\operatorname{Spec} B/\mathfrak{b}$ is induced by $\bar{\varphi}\colon A/\mathfrak{a} \to B/\mathfrak{b}$.

(iv) ($\star\star$) Take a prime ideal $\mathfrak{p}$ of A. Consider the multiplicatively closed system $S = A - \mathfrak{p}$ of A, and the ideal $S^{-1}\mathfrak{p} = \mathfrak{p}A_{\mathfrak{p}}$ in $S^{-1}A = A_{\mathfrak{p}}$. Let $B_{\mathfrak{p}} = S^{-1}B$ (and we always mean $\varphi(S)^{-1}B$, but this is cumbersome to write) and $\mathfrak{p}B_{\mathfrak{p}}$ be extension of $\mathfrak{p}A_{\mathfrak{p}}$ in $B_{\mathfrak{p}}$. Prove that the subspace $f^{-1}(\mathfrak{p})$ of $\operatorname{Spec} B$ is naturally homeomorphic to $\operatorname{Spec} B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}} = \operatorname{Spec} B \otimes_A k(\mathfrak{p})$, where $k(\mathfrak{p})$ is the residue field of the local ring $A_{\mathfrak{p}}$.

(v) Read (again) the definition of fibre of a morphism of schemes on page 89. ($\star\star$) If $f\colon Y \to X$ is a morphism of schemes and $x \in X$ a point, then the underlying topological space of the fibre $Y_x = Y \times_X \operatorname{Spec} k(x)$ is homeomorphic to $f^{-1}(x)$ with the induced topology. (The important step here is to reduce to the affine case, then use (iv)).

Separated morphism

(i) Read the definition of *separated morphism* on page 96. Make sure you understand the way the universal property of the fibred product is being used. Make sure you recall what a closed immersion is.

(ii) Read Example 4.0.1. ($\star$) How many open affine subsets do you need to cover X, the affine line with the origin doubled? ($\star$) How many affine open affine subsets do you need to cover $X \times_k X$? ($\star\star$) Why are all four origins in the closure of $\Delta(X)$?

(iii) Read the proof of Proposition 4.1. ($\star$) Why is $X \times_Y X$ given by $A \otimes_B A$? (If you do not remember it, then read again the first step of Theorem II.3.3!)

(iv) Read Corollary 4.2. ($\star$) What does it mean that it is a local question whether the map of sheaves $\mathcal{O}_{X \times_Y X} \to \Delta_* \mathcal{O}_X$ is surjective? ($\star$) Why is $U \times_V U$ open and affine?