Reading guidelines 3

June 10, 2017

1 Morphisms of finite type

(i) ($\star$) A closed immersion is of finite type. Careful: you will need the following fact,[1] which we will prove in the following lessons: if Y is a closed subscheme of an affine scheme $X = \operatorname{Spec} A$, then Y is also affine, and in fact Y is the closed subscheme determined by a suitable ideal $\mathfrak{a} \subseteq A$ as the image of the closed immersion $\operatorname{Spec} A/\mathfrak{a} \to \operatorname{Spec} A$. (Hint: thanks to this fact, you can even prove without more effort that a closed immersion is finite. Is $A/\mathfrak{a}$ an A-algebra which is a finitely generated A-module?)

For the next two exercises recall that being of finite type is a local property, i.e. a morphism $f\colon X \to Y$ is of finite type if and only if for *every* open affine subset $V = \operatorname{Spec} B$ of Y, $f^{-1}(V)$ can be covered by a finite number of open affines $U_j = \operatorname{Spec} A_j$, where each A_j is a finitely generated B-algebra. (We have not proved this fact yet.)

(ii) ($\star$) A composition of morphisms of finite type is of finite type. (Hint: this corresponds to the algebraic fact that if B is a finitely generated A-algebra and C is a finitely generated B-algebra, then C is a finitely generated A-algebra.)

(iii) ($\star$) Morphisms of finite type are stable under base extension. (Hint: this corresponds to the algebraic fact that if A is a finitely generated R-algebra and B is any R-algebra, then $A \otimes_R B$ is a finitely generated B-algebra.)

2 Separated morphisms

Use the valuative criterion of separatedness (Theorem II.4.3) to prove the following statements. Assume that all schemes are noetherian.

(i) ($\star$) A composition of separated morphisms is separated.

(ii) ($\star$) Separated morphisms are stable under base extension.

(iii) ($\star\star$) The product of two separated morphisms of schemes over a base scheme S is separated.[2] (Hint: use the valuative criterion or use (i) and (ii).)

(iv) ($\star$) If $f\colon X \to Y$ and $g\colon Y \to Z$ are two morphisms and if $g \circ f$ is separated, then f is separated.

(v) ($\star$) A morphism $j\colon X \to Y$ is separated if and only if Y can be covered by open subsets V_i such that $j^{-1}(V_i) \to V_i$ is separated for each i.

[1] This fact is (Ex. II.3.11(b)) or Corollary (II.5.10) in Hartshorne.

[2] The product of two morphisms $f\colon X \to Y$ and $f'\colon X' \to Y'$ of schemes over S is the unique morphism $X \times_S X' \to Y \times_S Y'$. ($\star$) Construct it using the universal property of the fibered product.

3 Proper morphisms

Read the (actual!) definition of *proper morphism* on page 100. Read Example 4.6.1: ($\star$) why is X separated and of finite type over k? ($\star$) Is the morphism $X \to \operatorname{Spec} k$ closed? ($\star$) Is it universally closed?
Use the valuative criterion of properness (Theorem II.4.7) to prove the following statements. Assume that all schemes are noetherian.

(i) ($\star$) A composition of proper morphisms is proper.
(ii) ($\star$) Proper morphisms are stable under base extension.
(iii) ($\star\star$) The product of two proper morphisms of schemes over a base scheme S is proper. (Hint: make sure you prove it is of finite type. Use (i) and (ii).)
(iv) ($\star$) If $f\colon X \to Y$ and $g\colon Y \to Z$ are two morphisms, if $g \circ f$ is proper, and if g is separated, then f is proper. (Hint: be careful about the uniqueness.)
(v) ($\star\star\star$) A morphism $j\colon X \to Y$ is proper if and only if Y can be covered by open subsets V_i such that $j^{-1}(V_i) \to V_i$ is proper for each i. (Hint: the difficult part is to prove that it is of finite type. Use (Ex. II.3.3).)

4 A closed immersion is (always) proper

We proved that a closed immersion $j\colon X \to Y$ is proper using the valuative criterion. Thus, we had to assume that X is noetherian. This hypothesis is unnecessary, as this series of exercises shows.

(i) ($\star\star$) If $j\colon X \to Y$ is a closed immersion, then the diagonal morphism $\Delta\colon X \to X \times_Y X$ is an isomorphism. (Hint: check that the universal property of the fibred product is satisfied by X. Existence is obvious, uniqueness is not.)
(ii) ($\star$) A closed immersion is separated. (Hint: use (i).)
(iii) ($\star$) A morphism $j\colon X \to Y$ is a closed immersion if and only if Y can be covered by open subsets V_i such that $j^{-1}(V_i) \to V_i$ is a closed immersion for each i.
(iv) ($\star\star$) Closed immersions are stable under base change. (Hint: use the previous point to reduce to the affine case. Use the results about surjectivity and morphisms of affine schemes from Reading guidelines 2.)
(v) ($\star$) A closed immersion is proper. (Hint: use (iv) in order to prove that it is universally closed.)

5 Valuation rings

Read the definitions of *valuation* and *valuation ring* on page 40. In what follows R is a valuation ring and K is its field of fractions. The valuation is denoted $v\colon K - \{0\} \to G$, where G is a totally ordered abelian group.[3]

(i) ($\star$) Prove that $v(1) = 0$, where 0 denotes the neutral element in G.
(ii) ($\star$) Prove that for every $x \in K - \{0\}$ either $x \in R$ or $x^{-1} \in R$.
(iii) ($\star$) Prove that $x \in R$ is a unit in R if and only if $v(x) = 0$.
(iv) ($\star$) Conclude that there exists a unique maximal ideal in R.

[3]Recall that the total order $\leq$ on G satisfies the following property: $g \leq g'$ implies $g + h \leq g' + h$ for every $g, g', h \in G$.

(v) ($\star$) Suppose that R is a *discrete* valuation ring (i.e. $G = \mathbb{Z}$). Prove that it is a principal ideal domain. (Hint: any ideal is generated by an element t of minimal valuation and contains *all* elements $s \in R$ such that $v(s) \geq v(t)$.)

(vi) ($\star\star$) Conclude that a discrete valuation ring has exactly two prime ideals.