# Reading guidelines 4

June 14, 2017

## 1 Sheaves of modules

Read the definitions related to sheaves of modules on page 109 and 110.

(i) ($\star$) Imitating the definition of sheaf of $\mathcal{O}_X$-modules, how would you define a *pre*sheaf of $\mathcal{O}_X$-modules? ($\star\star$) Prove that the sheafification of a presheaf of $\mathcal{O}_X$-modules is a sheaf of $\mathcal{O}_X$-modules.

(ii) ($\star$) Why is the kernel, cokernel and image of a morphism of $\mathcal{O}_X$-module again an $\mathcal{O}_X$-module?

(iii) Let $\mathcal{F}, \mathcal{G}$ be sheaves of abelian groups on a topological space $X$. ($\star$) For any open subset $U \subseteq X$, show that the set $\operatorname{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ of morphisms of the restricted sheaves has a natural structure of abelian group. ($\star\star$) Show that the presheaf $U \mapsto \operatorname{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ is a sheaf of abelian groups. ($\star$) How do you adjust these definitions if $(X, \mathcal{O}_X)$ is a ringed space and $\mathcal{F}, \mathcal{G}$ are $\mathcal{O}_X$-modules?

(iv) Read the definition of sheaf associated to an $A$-module $M$. Compare it to the definition of the structure sheaf $\mathcal{O}$ on $\operatorname{Spec} A$ (page 70). ($\star$) Prove that $\mathcal{O} = \widetilde{A}$.

(v) ($\star$) Read Proposition (5.1) and prove it by imitating the proof of (2.2).

(vi) Read Proposition (5.2). ($\star$) Saying that the map $M \mapsto \widetilde{M}$ is functorial (i.e., it is a functor of categories) means that if you have a morphism $M \to N$ of $A$-module, then you automatically get a morphism $\widetilde{M} \to \widetilde{N}$ of $\mathcal{O}_X$-modules: construct it using the definitions. It should be obvious that the identity of $A$-modules becomes the identity of $\mathcal{O}_X$-modules and the composition of morphisms of $A$-modules becomes the composition of morphisms of $\mathcal{O}_X$-modules. ($\star$) Saying that it is exact means that if you have a short exact sequence $0 \to M' \to M \to M'' \to 0$ of $A$-modules, then you get a short exact sequence $0 \to \widetilde{M'} \to \widetilde{M} \to \widetilde{M''} \to 0$. Prove it looking at the stalks. ($\star$) Prove (b) and (c) looking at the stalks. ($\star\star$) Prove (d) and (e) using the definitions (warning: this can be a bit cumbersome).

(vii) Read the definition of quasi-coherent and coherent sheaf of $\mathcal{O}_X$-modules.

(viii) Read Example 5.2.1. ($\star$) Why is $\mathcal{O}_X$ coherent on any scheme? (Hint: use (iv)).

(ix) Read Example 5.2.2. ($\star$) Why is $i_*\mathcal{O}_Y$ isomorphic to $\widetilde{A/\mathfrak{a}}$? (Hint: use (5.2d)).

## 2 Quasi-compactness and other properties

Look up the following definitions if you don't remember them: noetherian topological space, noetherian ring, noetherian scheme, quasi-compact topological space, quasi-compact scheme.

(i) ($\star$) Show that a topological space is noetherian if and only if every open subset is quasi-compact.

(ii) ($\star$) If $X = \operatorname{Spec} A$ is an affine scheme, prove that $X$ is quasi-compact. (Hint: take an open cover, consider the complements, recall that $\bigcap V(\mathfrak{a}_i) = V(\sum \mathfrak{a}_i)$.)

(iii) ($\star$) If $A$ is a noetherian ring, prove that the underlying topological space of $\operatorname{Spec} A$ is noetherian.

(iv) ($\star\star$) More generally, prove that if $X$ is a noetherian scheme, then its underlying topological space is noetherian. (Hint: use (iii) and reduce to prove that, in a topological space, the union of two noetherian open subsets is noetherian.)

(v) Let $k$ be a field and consider $A = k[x_1, x_2, \ldots]$, the ring of polynomials in infinitely many variables over $k$. ($\star$) Prove that $A$ is not a noetherian ring. ($\star$) Prove that the underlying topological space of $\operatorname{Spec} A$ is not a noetherian topological space.

(vi) Take $A = k[x_1, x_2, \ldots]$ as before. Consider the ideal $\mathfrak{a} = (x_1^2, x_2^2, \ldots)$. ($\star$) Is $A/\mathfrak{a}$ a noetherian ring? ($\star$) What are the prime ideals of $A/\mathfrak{a}$? ($\star$) Find an example of a non-noetherian scheme such that its underlying topological space *is* noetherian.

A morphism $f\colon X \to Y$ of schemes is *quasi-compact* if there is a cover of $Y$ by open affine subsets $V_i$ such that $f^{-1}(V_i)$ is quasi-compact for each $i$.

(vi) ($\star$) If $X$ is a noetherian scheme, then any morphism $f\colon X \to Y$ is quasi-compact.

(vii) ($\star\star$) Show that $f$ is quasi-compact if and only if for *every* open affine subset $V \subseteq Y$, $f^{-1}(V)$ is quasi-compact. (Hint: use (ii). Suppose $V_i = \operatorname{Spec} B_i$. Cover each $f^{-1}(V_i)$ with finitely many open affine subsets $\operatorname{Spec} A_{ij}$. Cover $V$ with finitely many distinguished open affine subsets $D(f_{ik})$, $f_{ik} \in B_i$. Prove that $f^{-1}(D(f_{ik}))$ is quasi-compact.)

(viii) ($\star\star$) Show that a morphism $f\colon X \to Y$ is of finite type if and only if it is locally of finite type and quasi-compact.

(ix) ($\star\star$) If $f\colon X \to Y$ and $g\colon Y \to Z$ are two morphisms, $f$ is quasi-compact, and $g \circ f$ is (locally) of finite type, then $f$ is of finite type. (Hint: cover $Z$ with open affine subsets $\operatorname{Spec} C$, so that $(g \circ f)^{-1}(\operatorname{Spec} C)$ is covered by $\operatorname{Spec} A_i$, with each $A_i$ a finitely generated $C$ algebra. For each point in $g^{-1}(\operatorname{Spec} C)$ choose an affine neighborhood $\operatorname{Spec} B$ and cover $f^{-1}(\operatorname{Spec} B) \cap \operatorname{Spec} A_i$ with distinguished affine subsets $D(a_{ij})$, $a_{ij} \in A_i$. Let $A_{ij}$ be $(A_i)_{a_{ij}}$. You have homomorphisms $C \to A_i \to A_{ij}$ and $C \to B \to A_{ij}$. Deduce that each $A_{ij}$ is a finitely generated $B$-algebra.)

# 3 Non-affine open subsets

(i) ($\star$) Let $k$ be a field and consider $\mathbb{A}^2_k = \operatorname{Spec} k[x,y]$. Prove that the open subset $U = \mathbb{A}^2_k \setminus \{(0,0)\}$ is not affine. (Hint: recall that for an integral scheme $X$ you can define a field $K = \mathcal{O}_\xi$, where $\xi$ is the generic point; this is called the *function field* of $X$. If $\operatorname{Spec} A$ is an open affine subset of $X$, then you can consider $A$ as a *subring* of $K$. In fact, $K$ is the quotient field of each $A$. In our situation $U$ contains both $\operatorname{Spec} k[x,y]_{(x)}$ and $\operatorname{Spec} k[x,y]_{(y)}$ (what are these open subsets?). If $U$ were affine...)

(ii) ($\star\star$) Let $X$ be a separated scheme over an affine scheme $S$. Let $U$ and $V$ be open affine subsets of $X$. Prove that $U \cap V$ is also affine. (Hint: prove that this diagram exists and is cartesian

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & X \arrow[d, "\Delta"] \\ U \times_S V \arrow[r] & X \times_S X \end{tikzcd}$$

then use the fact that closed immersions are stable under base change and the fact from RG3.)

(iii) ($\star$) Give an example to show that this fails if $X$ is not separated.