

Facts about (infinite) sets of numbers

- \mathbb{Z}, \mathbb{Q} are countable (there is a bijection among them and the integers \mathbb{N})
- \mathbb{R} is uncountable (no bijection with \mathbb{N} : Cantor's Diagonal Argument)
- $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$
- $|\mathbb{R}| = |\mathcal{P}(\mathbb{N})| = 2^{\aleph_0} = \aleph_1$
- Infinite hierarchy of cardinals...

Reals vs Rationals

Rationals

1/1	1/2	1/3	1/4	1/5	1/6	1/7	1/8	1/9	...
2/1	2/2	2/3	2/4	2/5	2/6	2/7	2/8	2/9	...
3/1	3/2	3/3	3/4	3/5	3/6	3/7	3/8	3/9	...
4/1	4/2	4/3	4/4	4/5	4/6	4/7	4/8	4/9	...
5/1	5/2	5/3	5/4	5/5	5/6	5/7	5/8	5/9	...
6/1	6/2	6/3	6/4	6/5	6/6	6/7	6/8	6/9	...
7/1	7/2	7/3	7/4	7/5	7/6	7/7	7/8	7/9	...
8/1	8/2	8/3	8/4	8/5	8/6	8/7	8/8	8/9	...
9/1	9/2	9/3	9/4	9/5	9/6	9/7	9/8	9/9	...
...

- Inf x Inf table
- Every **point** is one number
- **Finite** description of each number.

Reals

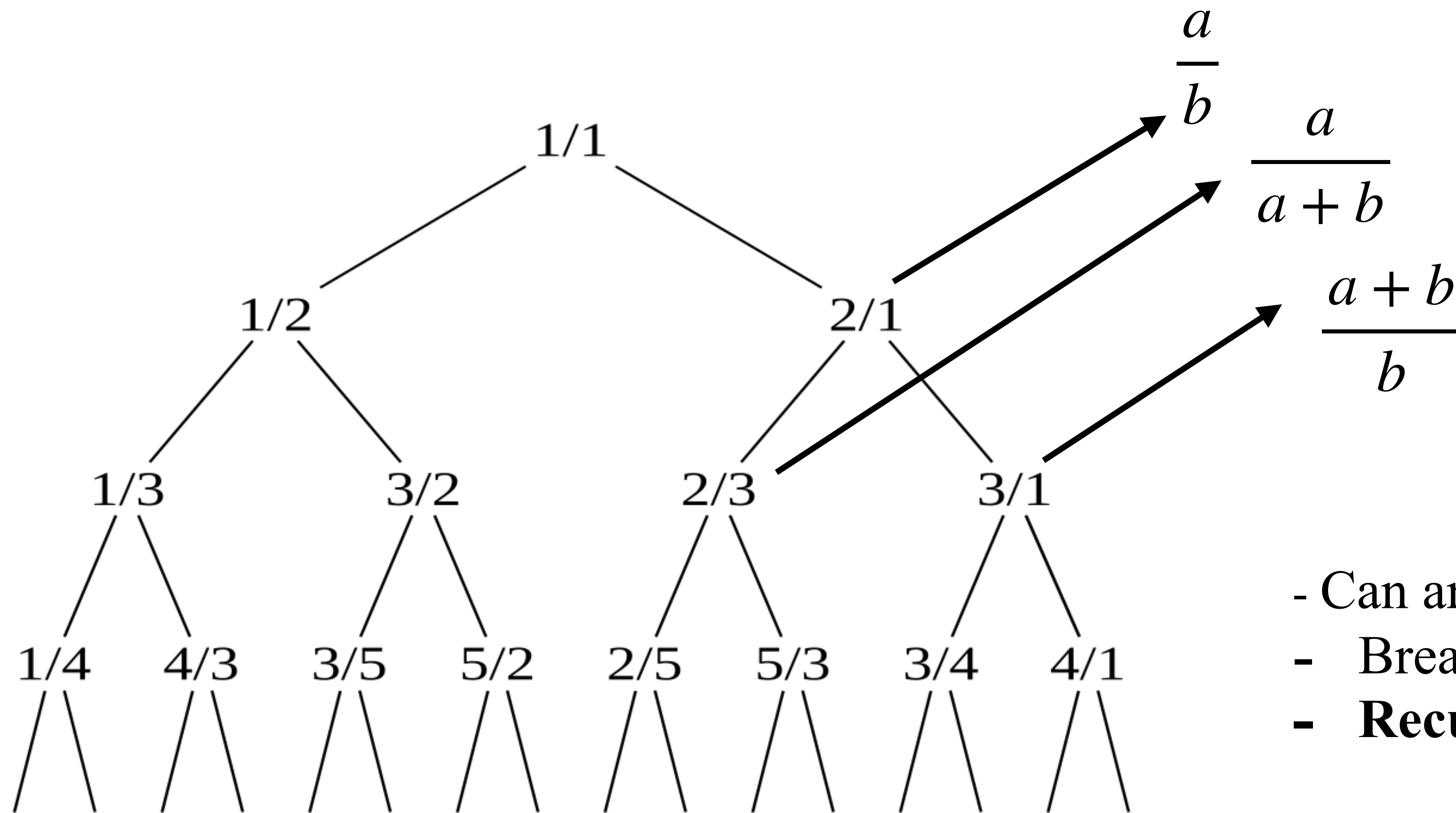
s_1	=	0	0	0	0	0	0	0	0	0	0	...	
s_2	=	1	1	1	1	1	1	1	1	1	1	...	
s_3	=	0	1	0	1	0	1	0	1	0	1	...	
s_4	=	1	0	1	0	1	0	1	0	1	0	...	
s_5	=	1	1	0	1	0	1	1	0	1	0	...	
s_6	=	0	0	1	1	0	1	1	0	1	1	...	
s_7	=	1	0	0	0	1	0	0	1	0	0	...	
s_8	=	0	0	1	1	0	0	1	1	0	0	1	...
s_9	=	1	1	0	0	1	1	0	0	1	1	0	...
s_{10}	=	1	1	0	1	1	1	0	0	1	0	1	...
s_{11}	=	1	1	0	1	0	1	0	0	1	0	0	...
:	:	:	:	:	:	:	:	:	:	:	:	...	

Wikipedia

$$s = 10111010011...$$

- Inf x Inf table
- Every **line** is one number
- **Infinite** description of each number.

Calkin-Wilf Tree Construction



- Can arrange rationals in a tree
- Breadth-first tree traversal = \mathbb{Q}^+
- **Recursion formula:**

$$q_{i+1} = \frac{1}{2\lfloor q_i \rfloor - q_i + 1}$$

Algorithm: Generate all the rationals

```
def calkin_wilf_positive():  
    a, b = 1, 1  
    while True:  
        yield (a, b)  
        k = a // b # floor division  
        a, b = b, 2 * k * b - a  
  
def all_rationals():  
    yield 0, 1 # zero  
    for a, b in calkin_wilf_positive():  
        yield a, b # positive  
        yield -a, b # negative  
  
gen = all_rationals()  
for _ in range(n):  
    a, b = next(gen)  
    print(f"{a}/{b}")
```

Uncountable Reals

Cannot devise something similar for reals!

For **countable** reals can work, e.g. :

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right).$$

No algorithm for uncountable reals.

Uncountable reals are **more** than countable ones.

Uncountable reals are sequences of digits which look **random** (no compressibility...)