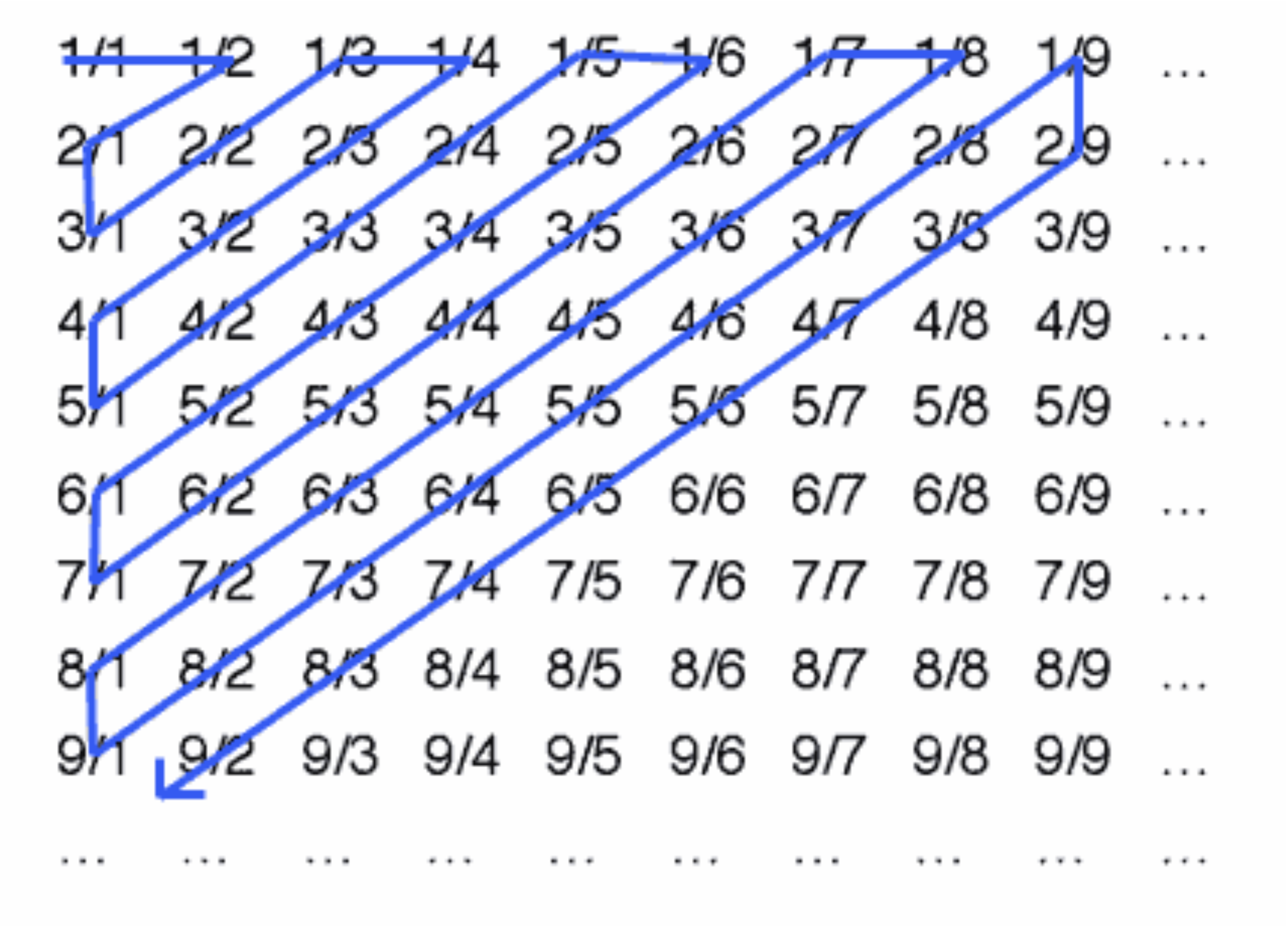
Facts about (infinite) sets of numbers

- $\mathbb{Z}, \mathbb{Q}$ are countable (there is a bijection among them and the integers $\mathbb{N}$)
- $\mathbb{R}$ is uncountable (no bijection with $\mathbb{N}$: Cantor's Diagonal Argument)
- $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$
- $|\mathbb{R}| = |\mathscr{P}(\mathbb{N})| = 2^{\aleph_0} = \aleph_1$
- Infinite hierarchy of cardinals…

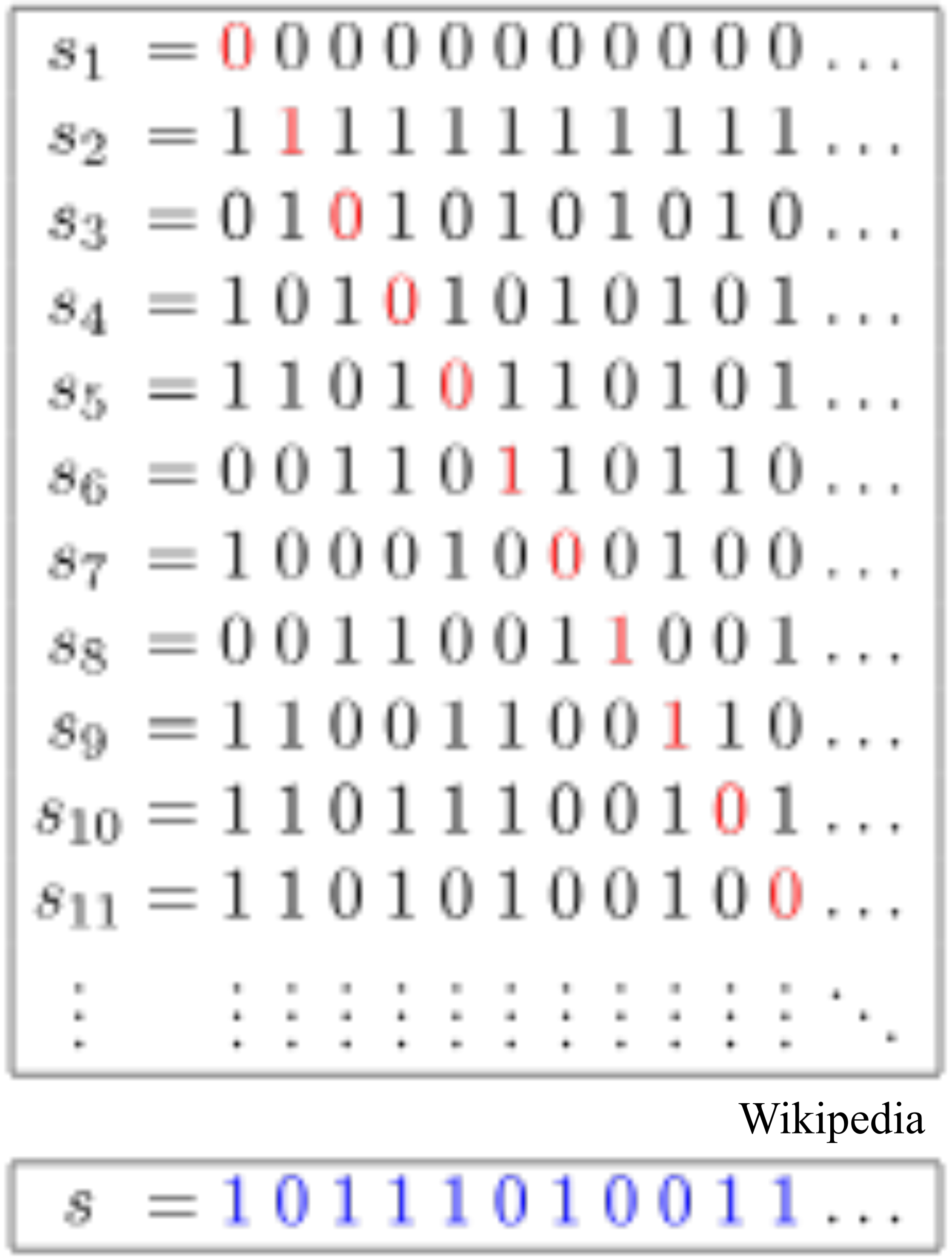
Reals vs Rationals

Rationals



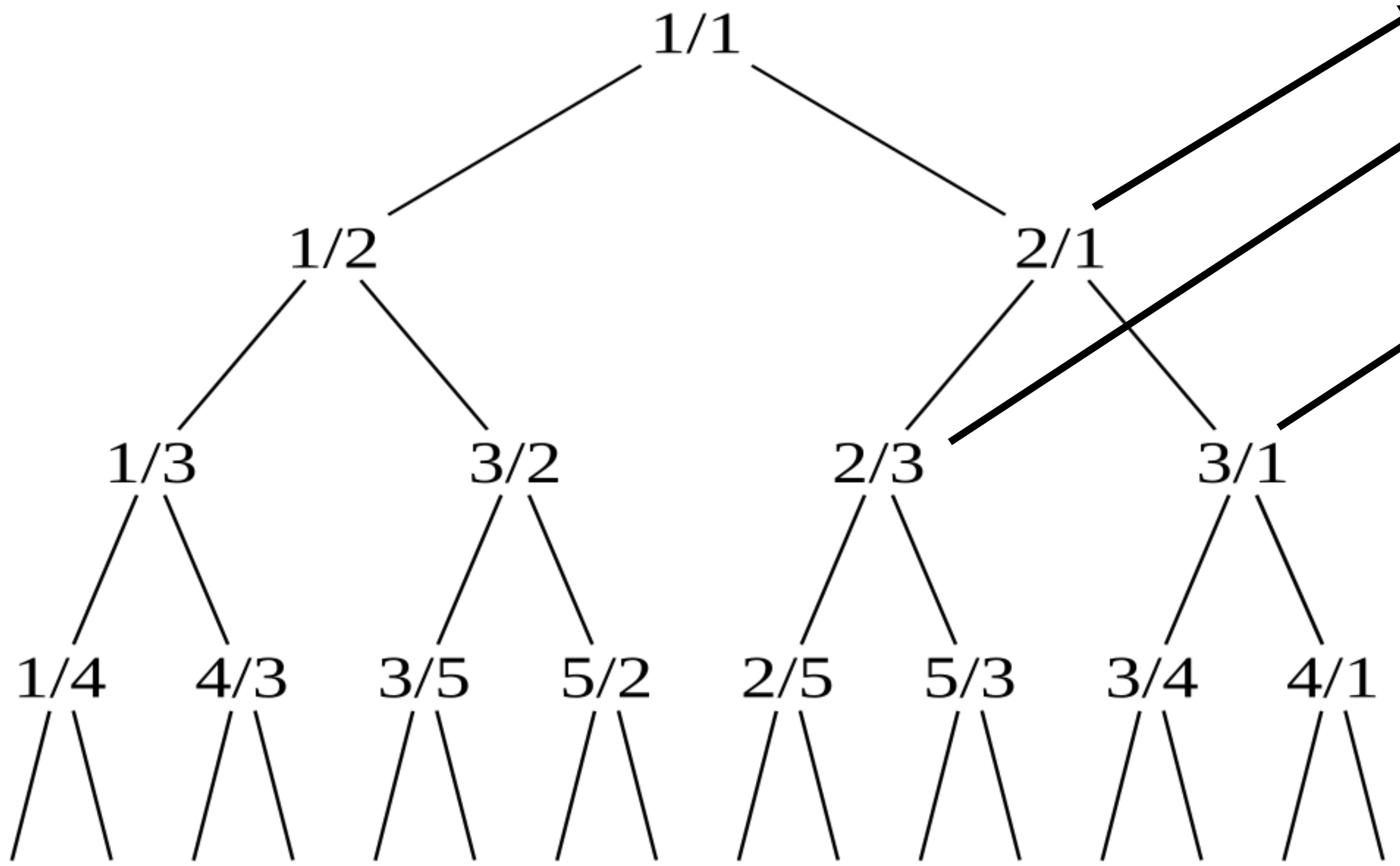
- Inf x Inf table
- Every point is one number
- Finite description of each number.

Reals



- Inf x Inf table
- Every line is one number
- Infinite description of each number.

Calkin-Wilff Tree Construction



Wikipedia

$$\frac{a}{b}$$

$$\frac{a}{a+b}$$

$$\frac{a+b}{b}$$

- Can arrange rationals in a tree
- Breadth-first tree traversal = $\mathbb{Q}^+$
- **Recursion formula:**

$$q_{i+1} = \frac{1}{2\lfloor q_i \rfloor - q_i + 1}$$

Algorithm: Generate all the rationals

```
def calkin_wilf_positive():
    a, b = 1, 1
    while True:
        yield (a, b)
        k = a // b # floor division
        a, b = b, 2 * k * b - a

def all_rationals():
    yield 0, 1 # zero
    for a, b in calkin_wilf_positive():
        yield a, b # positive
        yield -a, b # negative

gen = all_rationals()
for _ in range(n):
a, b = next(gen)
print(f"{a}/{b}")
```

Uncountable Reals

Cannot devise something similar for reals!

For countable reals can work, e.g. :

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right).$$

No algorithm for uncountable reals.

Uncountable reals are more than countable ones.

Uncountable reals are sequences of digits which look random (no compressibility…)