Canada's national laboratory
for particle and nuclear physics
and accelerator-based science

Topics across Physics and Finance

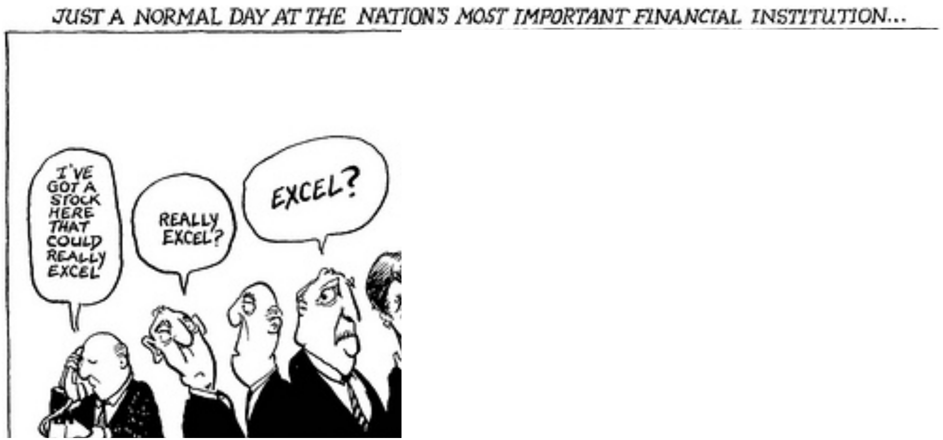
Luca Doria
TRIUMF

The intersections between Physics and Finance/Economy are many!
A very personal (and limited) selection of topics:

I) Introduction and Short History
II) Some Physics Problems and Probability Distributions
- Gaussian Distributions
- Non-Gaussian Distributions
- Examples from Physics

III) Stochastic Processes
IV) Stable Distributions
V) Stocks
VI) Derivates: Options
VII) Summary & Conclusions

Is there any logic behind Economics or the Stock Market?



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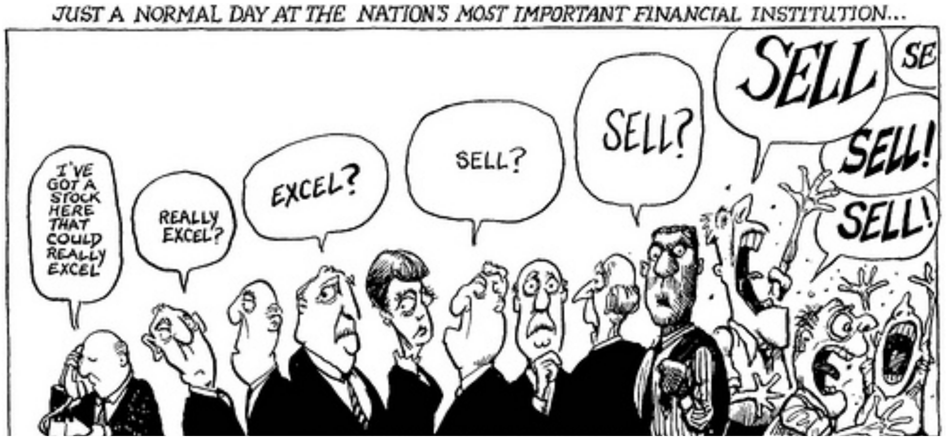

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Historical Overview

I) R. Brown (1773-1858): Observation of random motion in pollen samples
II) L. Bachelier (1870-1946): First attempt to model stocks movements
III) A. Einstein (1905): First model of the brownian motion
IV) Mandelbrot and Pareto Distributions
V) K. Ito and R. Stratonovich (~1950-60): Calculus with random variables
VI) Black, Merton, Scholes: Stochastic model for options (~1973)
1997 Nobel Prize in Economics


The Theory of Speculation
Louis Bachelier (1870-1946)


K. Ito (1915-2008)

Now:
"Econophysics": tries to apply physics methods to Finance and Economy:

- Stochastic Processes
- Statistical Physics
- Agent-based Models
- Statistical Analysis
- Feynman's Path Integrals

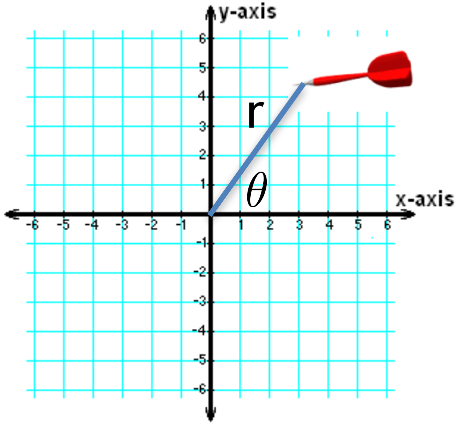

R. Black, M. Scholes


B. Mandelbrot (1924-2010)

PART 1:

PHYSICS MODELS and PROBABILITY DISTRIBUTIONS: EXPONENTIALS vs POWER LAWS

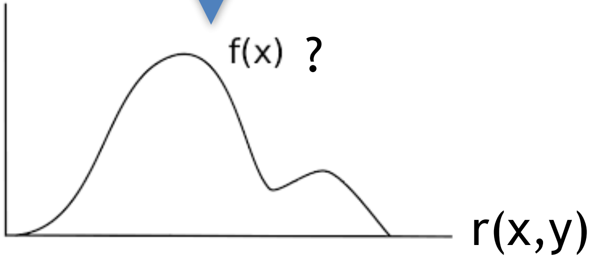
Probability Distributions



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\begin{cases} dx/d\theta = -r\sin\theta \\ dy/d\theta = r\cos\theta \end{cases}$$

$$\frac{dx/d\theta}{dy/d\theta} = -\frac{y}{x} \qquad [1]$$



Probability Distributions

Distribution of the darts: $g(r) = f(x,y) = h(x)k(y)$

Hypothesis: Distribution independent from the angle (neglect e.g. gravity) $\dfrac{\partial g}{\partial \theta} = 0$

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Substituting [1] (see prev. slide) $\underbrace{\dfrac{1}{h(x)}\dfrac{\partial h}{\partial x}\dfrac{1}{x}}_{=X} = \underbrace{\dfrac{1}{k(y)}\dfrac{\partial k}{\partial y}\dfrac{1}{y}}_{=X} \quad \Rightarrow \quad f(x,y) \propto e^{X(x^2+y^2)}$

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QUESTION: Is X positive or negative?

Why Gaussians appear so often?

The Central Limit Theorem (Lindenberg, Levy ~1920)

Consider N independent random variables X_i (i=1..N) drawn from the same probability distribution f such that :

$$S_N = \frac{X_1 + X_2 + .. + X_N}{N} \to \mu \qquad N \to \infty \qquad \text{“law of large numbers”}$$

$$Var(f) = \sigma^2 \qquad 0 < \sigma^2 < \infty$$

Then:

$$\sqrt{N}\left(\frac{1}{N}\sum_{i=1}^{N} X_i - \mu\right) \to \mathcal{N}(0, \sigma^2) \qquad N \to \infty$$

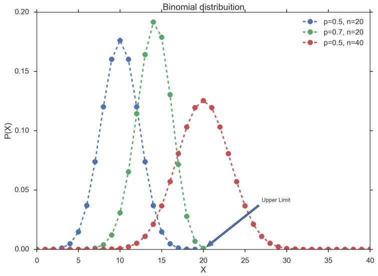
The Gaussian as a limit distribution

Binomial Distribution

$$P(k,N) = \frac{N!}{(N-k)!k!}p^k(1-p)^{N-k} \underset{N\to\infty}{\longrightarrow} \mathcal{N}(Np, Np(1-p))$$

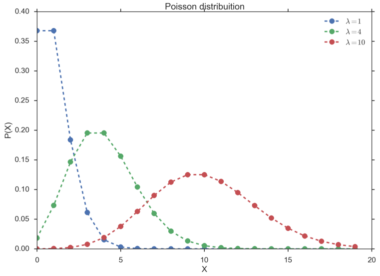
$$Np \to \infty$$

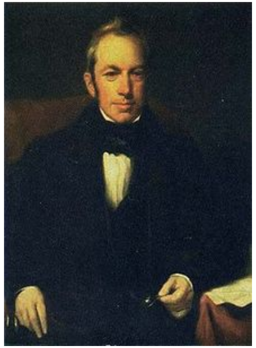
$$N(1-p) \to \infty$$



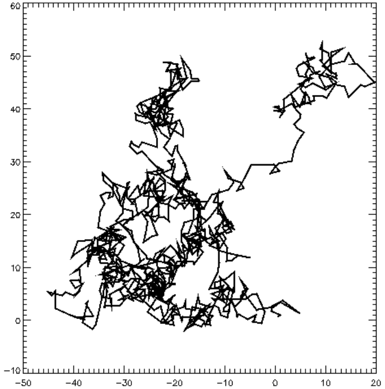
Poisson Distribution

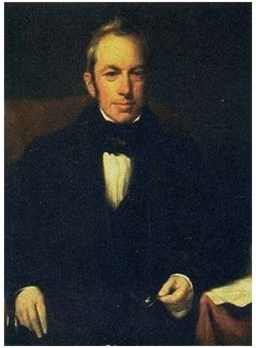
$$P(k) = \frac{\lambda^k e^{-\lambda}}{k!} \underset{\lambda\to\infty}{\longrightarrow} \mathcal{N}(\lambda, \sqrt{\lambda})$$





R. Brown (1773-1858): Observation of random motion in pollen samples





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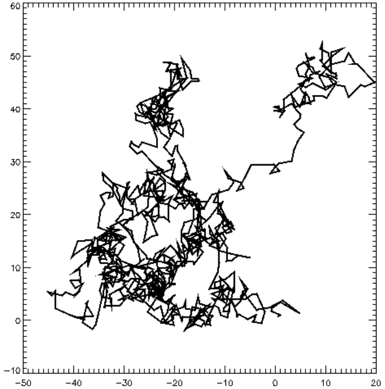


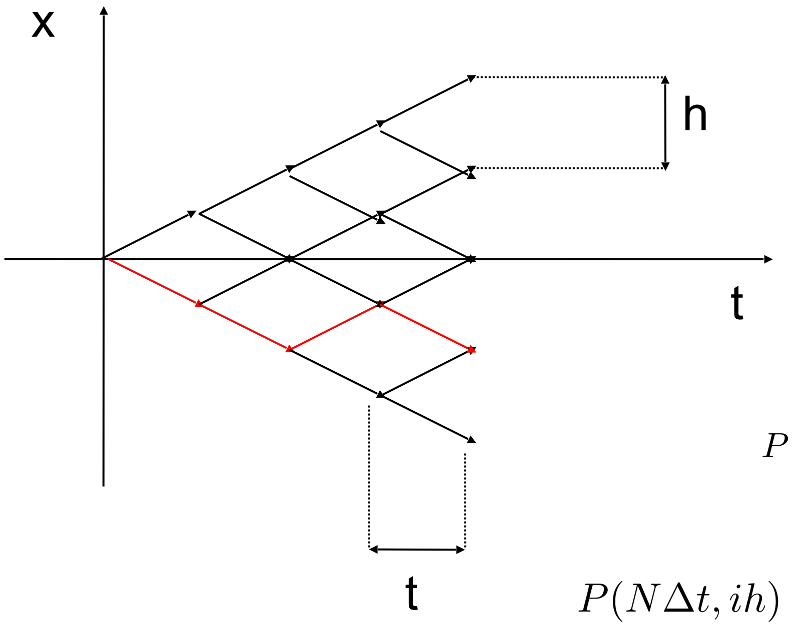
A. Einstein (1905): First quantitative explanation of the brownian motion

$$\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} = 0 \qquad D = \frac{RT}{N}\frac{1}{6\pi k r} \qquad \sqrt{< \Delta x^2 >} = \sqrt{2Dt}$$

Opportunity to estimate N_A (check of the atomic hypothesis) !

"… It is hoped that some enquirer may succeed shortly in solving the problem suggested here, which is so important in connection with the theory of heat."
Berne, May 1905 (Received, 11 May 1905).

A Simple Model of the Brownian Motion



$$< \Delta x >= 0$$

$$Var(\Delta x) =< (\Delta x)^2 > -(< \Delta x >)^2 = \frac{1}{2}h^2 - \frac{1}{2}(-h)^2 = h^2$$

$$t = N\Delta t \Rightarrow Nh^2 = \frac{th^2}{\Delta t}$$

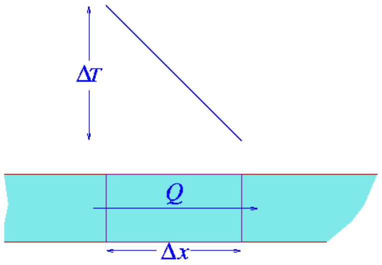
$$\sigma^2 t \Rightarrow h = \sigma\sqrt{\Delta t}$$

$$P((N+1)\Delta t, ih) = \frac{1}{2}P(N\Delta t, (i-1)h) + \frac{1}{2}P(N\Delta t, (i+1)h)$$

$$P(N\Delta t, ih) + \frac{\partial}{\partial t}P(N\Delta t, ih)\Delta t \quad P((N+1)\Delta t, ih) + \frac{h^2}{2}\frac{\partial^2}{\partial x^2}P(N\Delta t, ih)$$

$$\frac{\partial P}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 P}{\partial x^2} \quad \Rightarrow \quad P(x,t) = \frac{1}{\sqrt{2\pi(t-s)}}e^{-\frac{(x-x_0)^2}{2\sigma^2(t-s)}}$$

The Propagation of Heat



Fourier's Law: $q = -k\nabla u$

Thermodynamics: $Q = c_p \rho u$

If a change in internal energy in a material is given only by the heat flux across the boundaries in a space/time region:

$$x - \delta x \leq \xi \leq x + \delta x \qquad t - \delta t \leq \tau \leq t + \delta t$$

$$k \int_{t-\delta t}^{t+\delta t} \left[\frac{\partial u}{\partial x}(x + \delta x, \tau) - \frac{\partial u}{\partial x}(x - \delta x, \tau) \right] d\tau = k \int_{x-\delta x}^{x+\delta x} \int_{t-\delta t}^{t+\delta t} \frac{\partial^2 u}{\partial \xi^2} d\xi d\tau$$

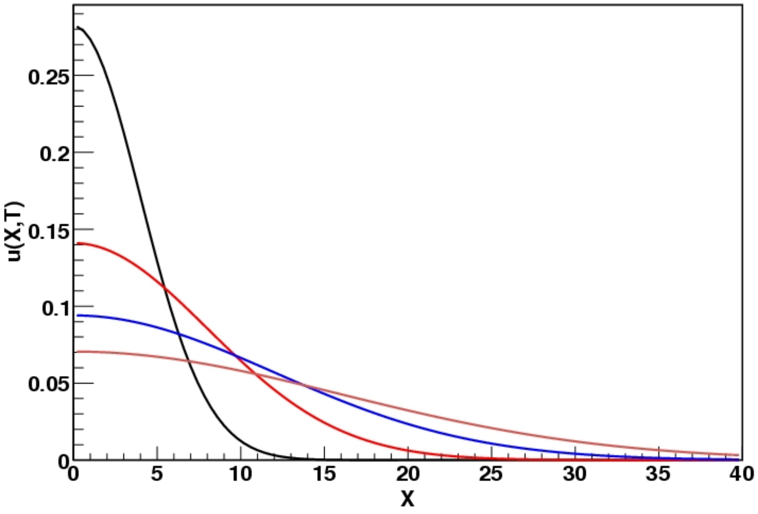
$$c_p \rho \int_{x-\delta x}^{x+\delta x} [u(\xi, t + \delta t) - u(\xi, t - \delta t)]\, d\xi = c_p \rho \int_{x-\delta x}^{x+\delta x} \int_{t-\delta t}^{t+\delta t} \frac{\partial u}{\partial \tau} d\xi d\tau$$

By Energy Conservation: $\dfrac{\partial u}{\partial t} - \dfrac{k}{c_p \rho} \dfrac{\partial^2 u}{\partial x^2} = 0$

Solutions of the Heat Equation

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = 0$$

Propagator $$G(x,t) = \frac{1}{\sqrt{4\pi k t}} e^{-\frac{x^2}{4kt}}$$



General Solution $$u(x,t) = \int G(x-y,t) f(y,t=0) dy$$

Are Gaussians really everywhere?
Are we really living in a Gaussian World?

WHY caring about non-gaussian distributions?

I) Random processes often encountered in physics (Probability/Distributions).

III) Not all random processes have gaussian tails !

IV) "Fat" tails are not always easy to explain: associated with "complex" systems, phase transitions, critical states, scale invariance, …

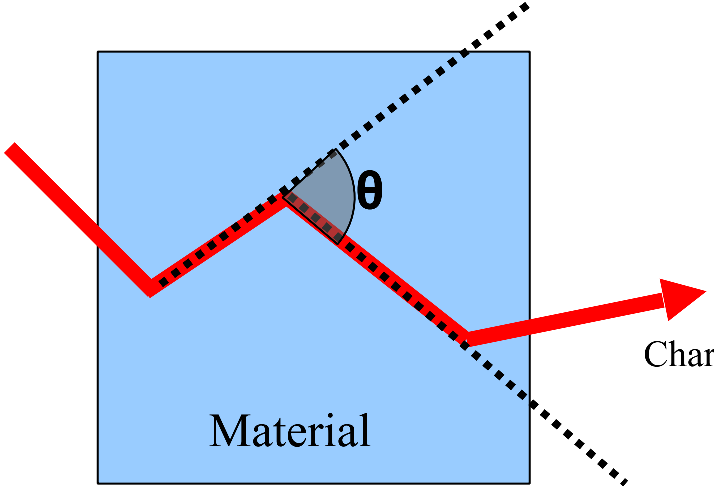
In general: something interesting for a physicist !

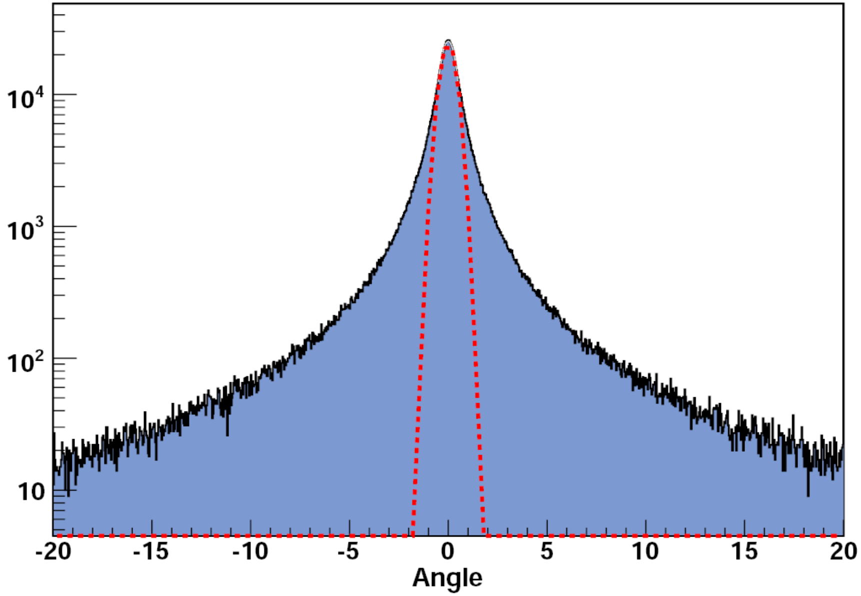
V) More formal methods needed to treat stochastic processes.

An example from Experimental Physics

Used e.g. in:
- Detector Physics and Simulation
- Medical Physics

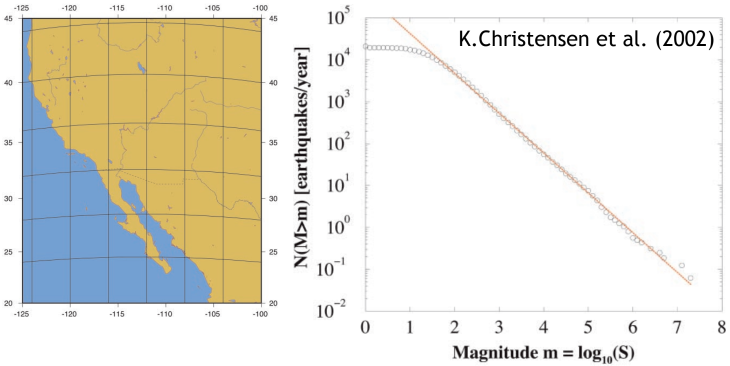
Two physics mechanisms:
- EM Force
- Strong Force



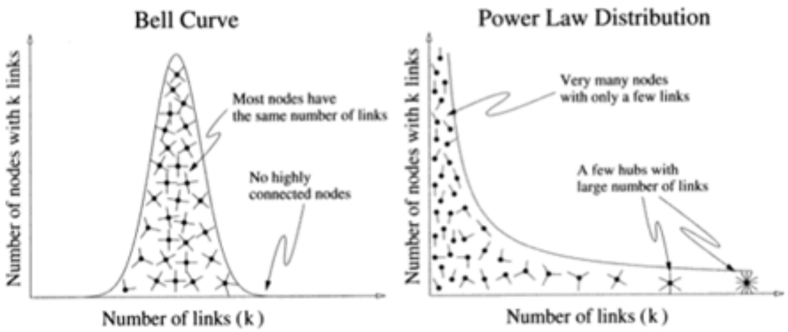


Other Examples

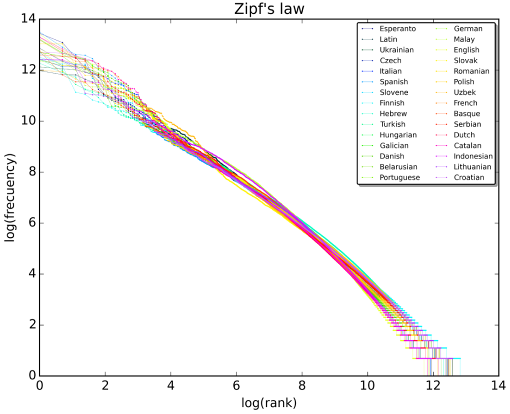
Distribution of the magnitude of earthquakes



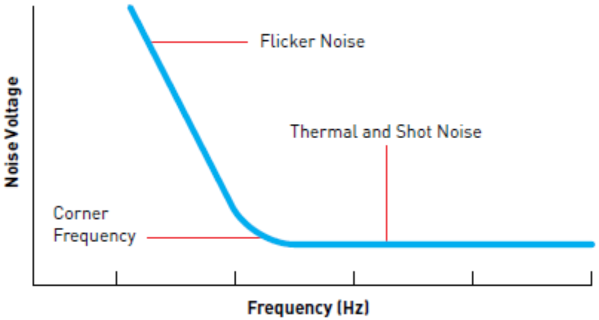
Connectivity in real networks



Zipf's Law in real languages



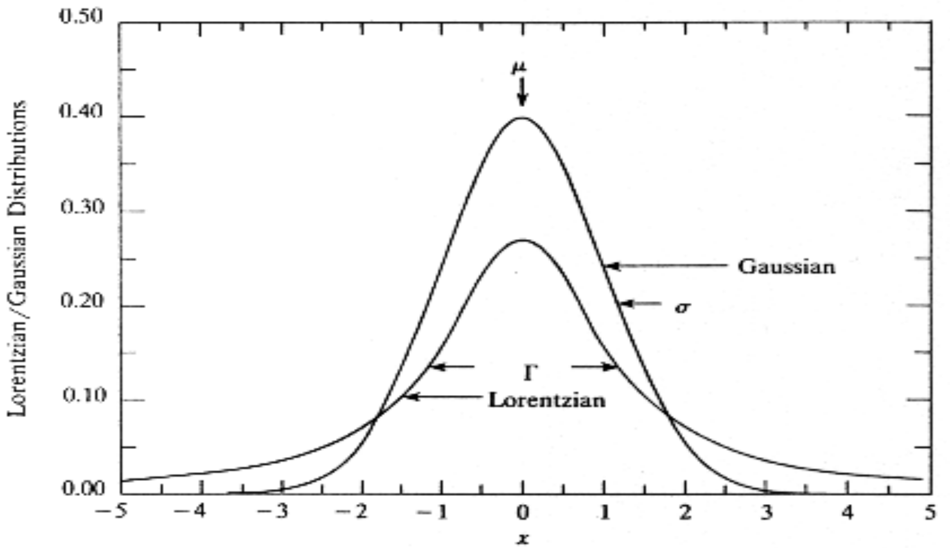
Pink Noise



The Lorentzian Distribution

Gaussian: $$P(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Lorentzian: $$P(x) = \frac{1}{\pi} \frac{\gamma}{(x-\mu)^2 + \gamma^2}$$



For the Lorentz distribution, in contrast to the Gaussian:
- The moments are not defined (infinite variance!).
- The Central Limit Theorem does not hold.

Power Laws are “Scale Free”

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Scale free distribution: $\rho(ax) = f(a)\rho(x)$

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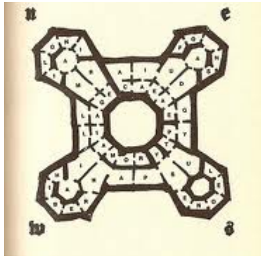
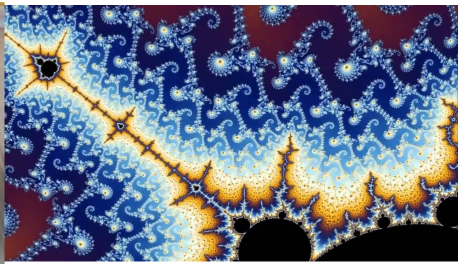
Interesting connections with:
fractals, phase transitions, criticality…





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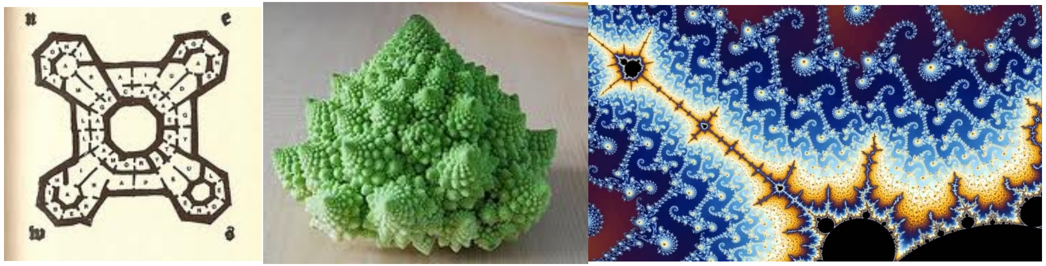
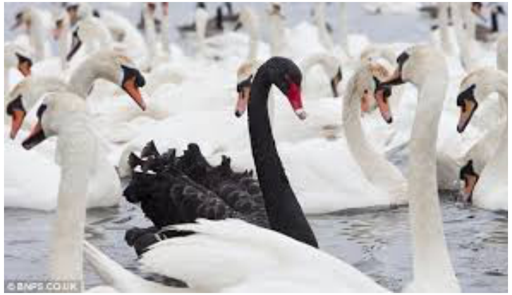




Black swans do exist …. !

Taleb, Nassim Nicholas (2010). *The Black Swan: the impact of the highly improbable*

Beyond the CL Theorem: Stable Distributions

Definition: x is a stable random variable if for every positive A,B exist C,D so that:

$$(Ax_1 + bx_2) \approx Cx + D$$

where x1 and x2 are copies of x.

Theorem: The sum of independent, identically distributed random variables (iid) converge to a stable distribution as the number of iid variables tends to infinity.

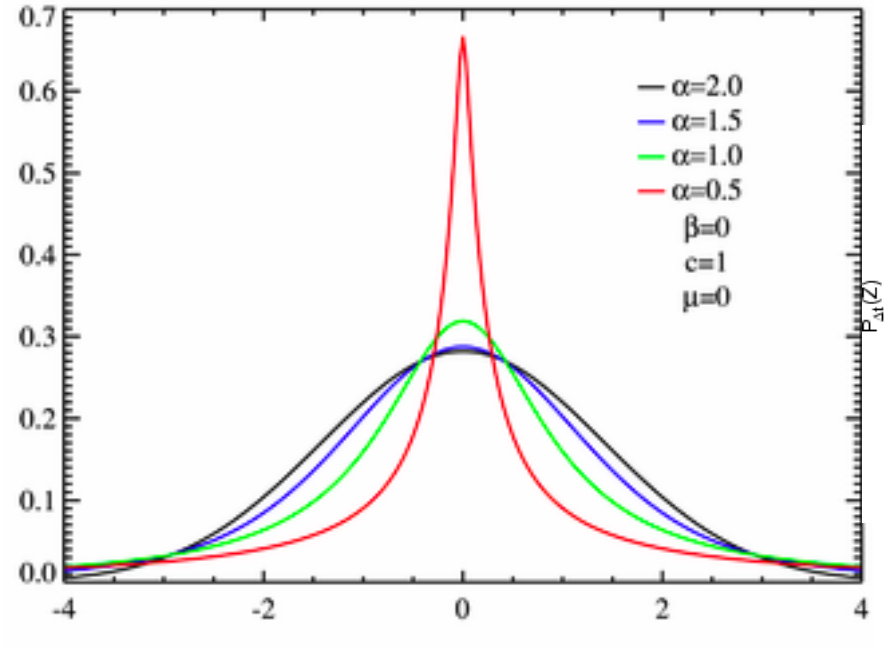
Stable (Levy) Distributions $S_\alpha(\gamma, \beta, \mu)$

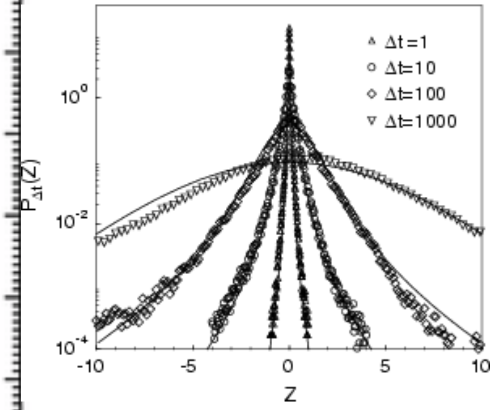
$$\ln\phi(k) = \begin{cases} i\mu k - \gamma|k|^\alpha \left[1 - i\beta\frac{k}{|k|}\tan(\pi/2\alpha)\right] & \alpha \neq 1 \\ i\mu k - \gamma|k|\left[1 + i\beta\frac{k}{|k|}(2/\pi)\tan(|k|)\right] & \alpha = 1 \end{cases}$$

$0 < \alpha < 2$	Stability Index
$\gamma > 0$	Scale Factor
$\mu \in \mathcal{R}$	Shift Factor
$\beta \in (-1, 1)$	Asymmetry Factor

$\alpha = 2$	Gaussian
$\alpha = 1; \beta = 0$	Lorentz

Levy Distributions





$$\lim_{x\to\infty} P(X > x) \approx \frac{\Gamma(1+\alpha)\sin(\pi\alpha/2)}{\pi|x|^{1+\alpha}} \approx \frac{1}{|x|^{1+\alpha}}$$

Levy Distributions have “fat tails”

Stocks and the Stock Market:

- Statistical Properties
- Stochastic Calculus

Many financial products are traded on the Stock Market.
We will take a look at:

- Stocks
- Derivates, in particular Options.

Questions:

- What are the statistical properties of the Stock Market?
- Are these properties universal?
- If yes, what is the cause?
- What are the best mathematical tools to tackle these problems?

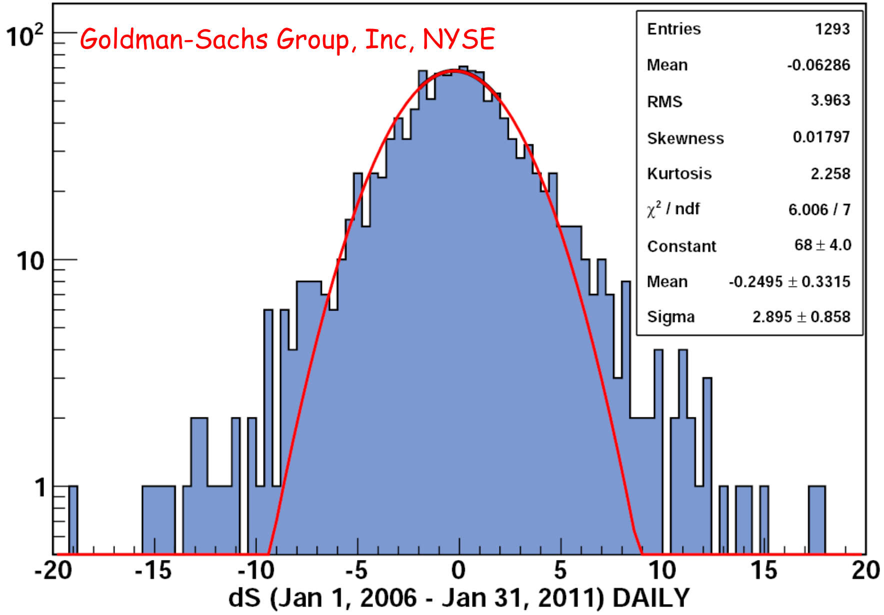
We will briefly describe two methods:

1) Statistical Analysis / Signal Analysis
2) Stochastic Calculus

Stocks

How to analyze/model these data?

The Distribution of Returns



Not Gaussian!

$$P_i(t)$$

$$R_i(t, \Delta t) = \ln P_i(t + \Delta t) - \ln P_i(t)$$

$$r_i(t, \Delta t) = \frac{R_i - \langle R_i \rangle}{\sigma_i}$$

$$\sigma_i(t, \Delta t) = \sqrt{\langle R_i^2 \rangle - \langle R_i \rangle^2}$$

Stochastic Calculus (Ito, Stratonovich)

$$dS = F(S,t)dt + G(S,t)dW \quad \Rightarrow \quad S(t) = S_{t_0} + \int_{t_0}^{t} F(S,\tau)d\tau + \int_{t_0}^{t} G(S,t)dW_\tau$$

$$dS = \sigma^2 dW$$ Brownian Process

$$dS = \mu S dt + \sigma S dW$$ Geometric Process

$$dS = \lambda(\mu - S)dt + \sigma S dW$$ “Return to the Mean” Process

$$dS = -\lambda S dt + \sigma dW$$ Ornstein-Uhlenbeck Process (fluctuation around zero)

An example of Ito's integration of a random variable W:

$$\int_{t_0}^{T} W_t dW_t = \frac{1}{2}(W_T^2 - W_{t_0}^2) - \frac{1}{2}(T - t_0)$$

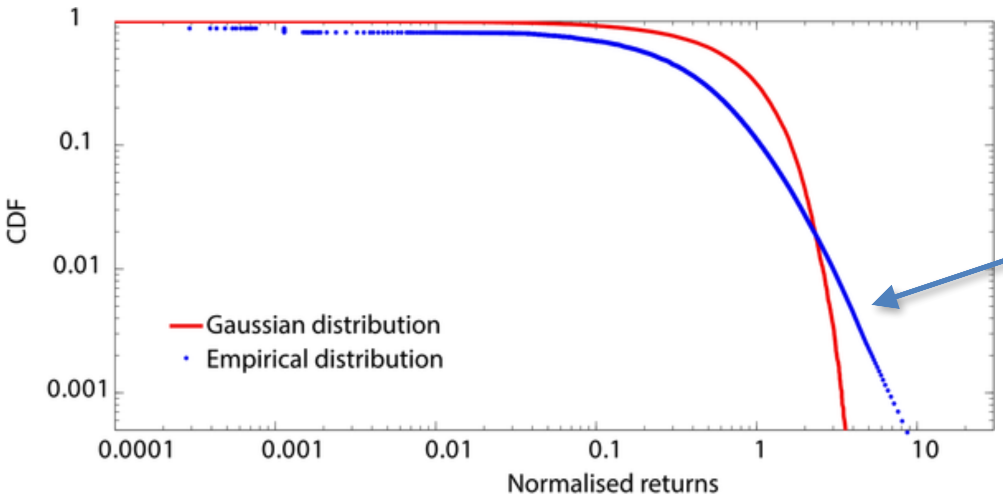
The most important formula: **Ito's Formula** for the differential:

If a random process is described by: $dS = F(S,t)dt + G(S,t)dW$

then a function Y of S and the time t has the following differential:

$$dY_t = \{\partial_t Y + F\partial_S Y + \frac{1}{2}G^2\partial_x^2 Y\}dt + G\partial_x Y dW_t$$

Cumulative Distribution



The distribution is not gaussian.
Close to a -3 power law.

$$r_i(t, \Delta t) = \frac{R_i - \langle R_i \rangle}{\sigma_i}$$

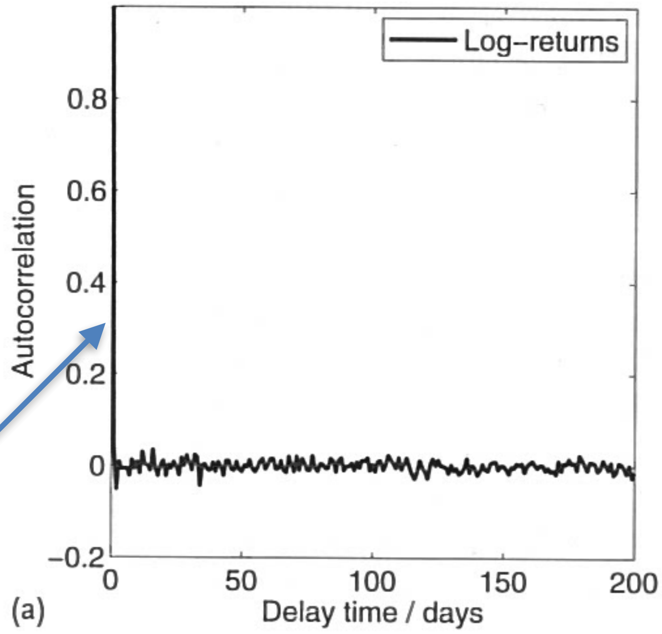
At small timescales (~min) the returns' distribution is not gaussian.
It has fat and asymmetric tails (2.8 and 3 for example).

The "inverse cubic law" is observed in many markets: what is its origin?

Autocorrelation

Autocorrelation of the returns:

$$\rho(T) = \langle r_\tau(t+T) r_\tau(t) \rangle$$

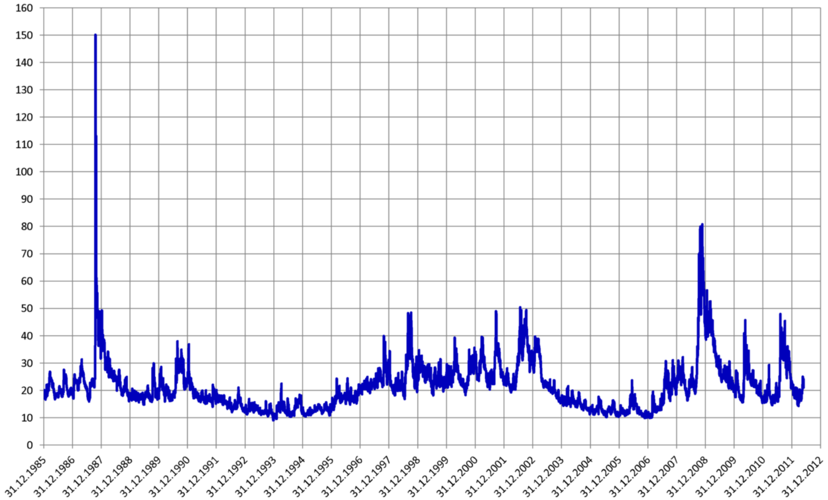


Very fast decay observed

Volatility: Standard deviation of the logarithmic returns:

$$V = \sqrt{\left[\sum_i^N \left(\ln \frac{S_i}{S_{i-1}}\right)^2 - \frac{1}{N}\left(\ln \frac{S_i}{S_{i-1}}\right)^2\right] \frac{1}{N-1}}$$

Volatility Clustering:
Contrary to the returns, volatility displays more "memory": small volatilities tend to be followed by small volatilities and large volatilities by large volatilities.



The “Stylized Facts”

Extensive data analysis on the distributors of stock markets can be briefly summarized in the following “stylized facts”:

1) The returns’ distribution exhibits fat and slightly asymmetric tails at short time scales (~min). It becomes slowly a gaussian at large (>day, month) time scales. This signals a non trivial time structure for the underlaying stochastic process.

2) No linear autocorrelation: the autocorrelation decays very quickly to zero on short (~min) time scales. This supports the “efficient market” hypothesis.

3) Volatility clustering. The volatility has more “memory” (or correlation) that the returns and therefore tends to cluster in time. Similar phenomena are observed in the velocity distributions of turbulent flows.

Many other observables: Power spectral density, Fourier analysis, …

Derivatives:

-Options

-....

Options

Robert C. Merton

Myron S. Scholes

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 1997 was awarded jointly to Robert C. Merton and Myron S. Scholes "for a new method to determine the value of derivatives "

Black, F. och M. Scholes, 1973, "The Pricing of Options and Corporate Liabilities", Journal of Political Economy, Vol. 81, pp. 637-654.

Black, F., 1989, "How We came Up with the Option Formula", The Journal of Portfolio Management, Vol. 15, pp. 4-8

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Options

Definition:

In finance, an option is a derivate financial instrument that establishes a contract between two parties concerning the buying or selling

of an asset at a reference price.

"Put" ("Call") European Options:

The owner of the option has the right (but no obligation) to sell (buy) a certain good at a certain time for a certain price.

Other jargon terms:

Underlying (Asset): A stock for a simple option (can be any good).

Strike Price: Price for buying or selling the underlying.

Expiration Date: When the option might be used.

American Options: The option can be used at any time before expiration.

Exotic Options: All other kind of options (e.g. asian options, ...).

Hypoteses:

I) The short-term interest rate r is known and constant and it is possible to ask for money at that rate.

II) The strike price X is known and constant

III) The stock price S follows a geometric stochastic process:

$$dS = \mu S dt + \sigma S dW$$

Where W is a Wiener process, μ is the expected return, σ the volatility.

IV) The stock does not pay dividends

V) No transaction costs and no limits to the short-selling

VI) There is no arbitrage

Derivation of the BS Equation

Build a portfolio made by stocks and options: $V = S - \frac{1}{\Delta}c \quad \overset{\text{variation}}{\Longrightarrow} \quad dV = dS - \frac{1}{\Delta}dc$

Apply Ito's Lemma to c(t,S): $$dc = \left[\frac{\partial c}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c}{\partial S^2}\right] dt + \frac{\partial c}{\partial S}\sigma dW$$

Substituting: $$dV = -\frac{1}{\Delta}\left[\frac{\partial c}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c}{\partial S^2}\right] dt$$

Equivalence to a risk-free portfolio:
(no arbitrage!). $$dV = -\frac{1}{\Delta}\left[\frac{\partial c}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c}{\partial S^2}\right] dt = \left(S - \frac{1}{\Delta}c\right) r dt$$

After some algebra: $$\frac{\partial c}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c}{\partial^2 S} = rc - rS\frac{\partial c}{\partial S}$$

A Physics Analogy

Example: dispersion of a pollutant in a river:

$$\frac{\partial c}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c}{\partial^2 S} = rc - rS\frac{\partial c}{\partial S}$$

Diffusion Convection Reaction

Dispersion: diffusion Term
Water Flow: Convection Term
Absorption (e.g. by sand): Reaction Term

But at the end....the BS Equation IS the heat equation!

Solutions

Solution: $$c(S,t) = SN(d_1) - Xe^{-r(T-t)}N(d_2)$$

with: $$d_1 = \frac{\ln\frac{S}{X} + (r + 1/2\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

$$N(d_i) = \int_{-\infty}^{d_i} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$$

Another technique: Montecarlo Simulations

The BS equation has an analytic solution.
Analytic solutions are possible in other “simple” cases, e.g.:

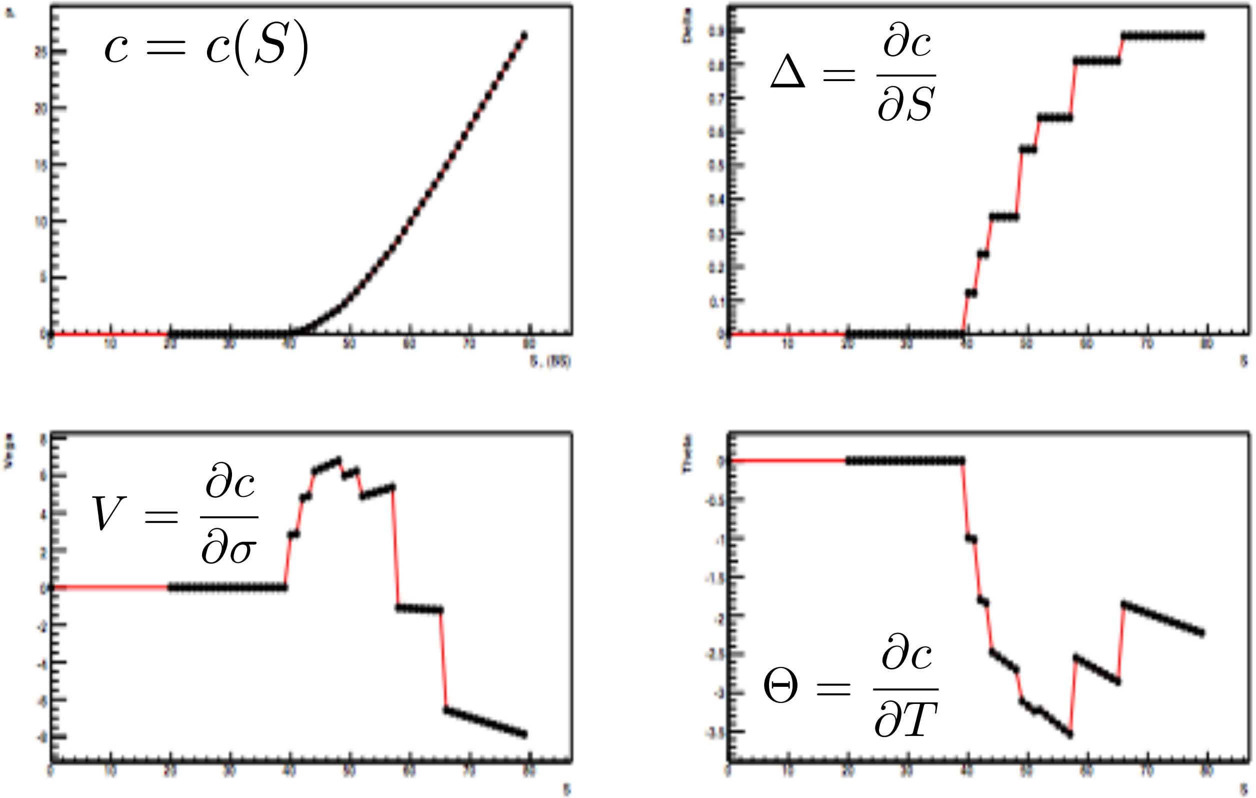
What about more complicated cases, like e.g:
Changing volatility
Stochastic processes different from the geometric one

A possibility is to use **Montecarlo simulations**.

Idea: Simulate N “paths” for the stock price and average at the end the reached price.

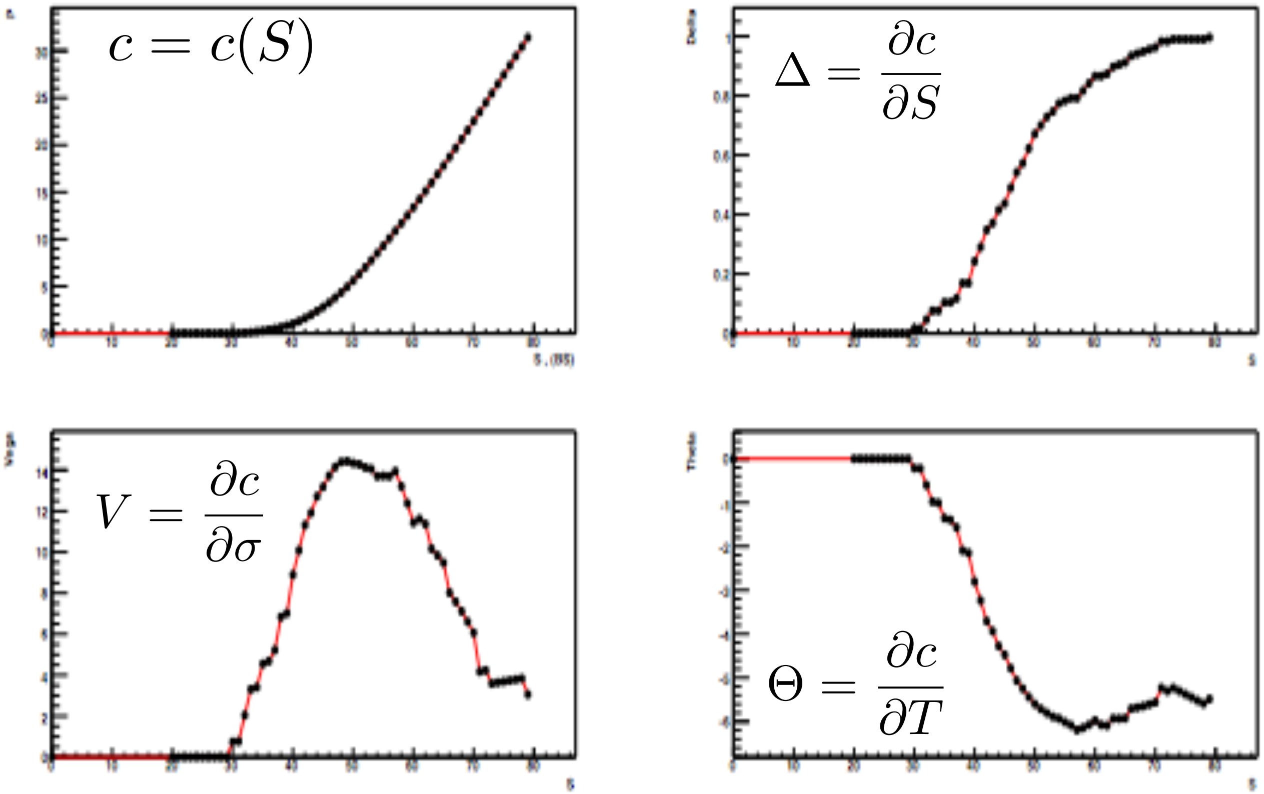
A MonteCarlo Solution: “The Greeks”

N=10



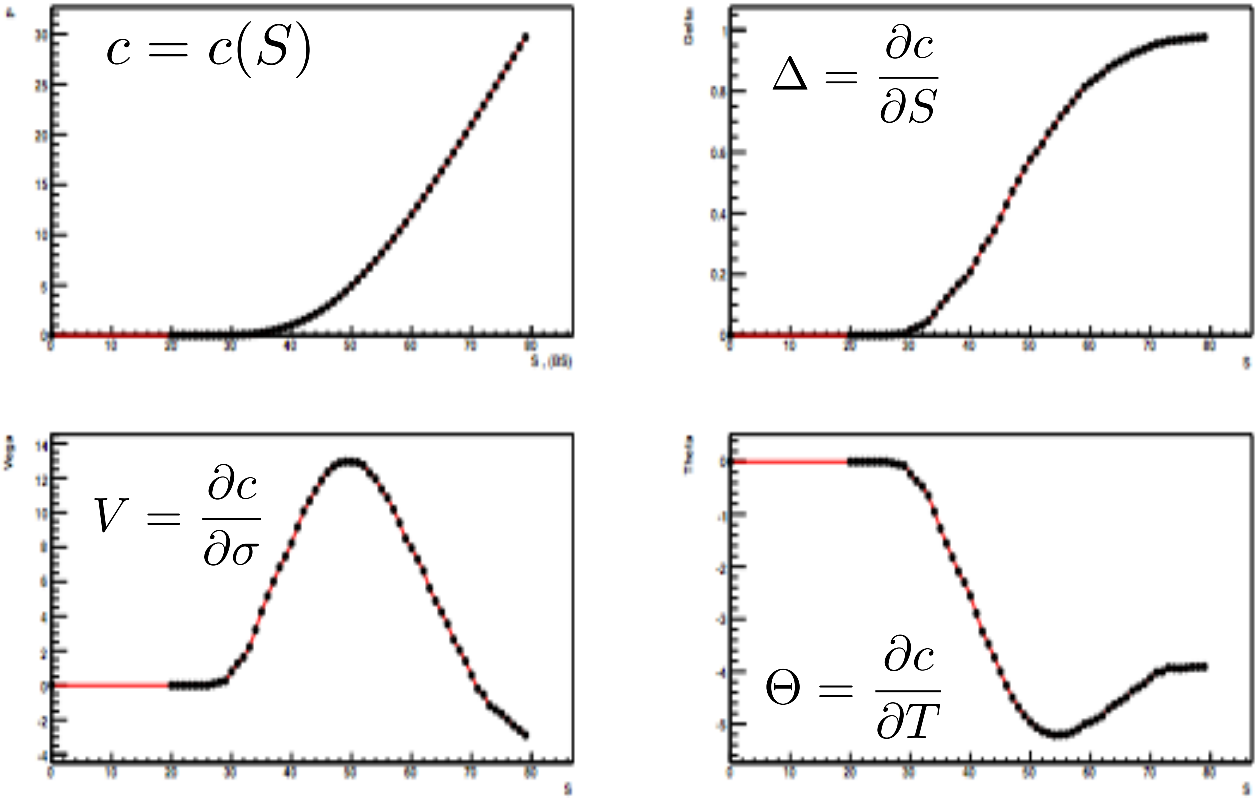
A MonteCarlo Solution: “The Greeks”

N=100



A MonteCarlo Solution: “The Greeks”

N=1000



Summary and Outlook

- Non-gaussianity (or “fat tails”) are relevant in the description of many
- natural phenomena.

- Stock markets display highly non trivial statistical properties.

- Analysis possible today: large data and computing power.

- Not only stocks: there are many financial instruments which can be analyzed with mathematical techniques common in physics. Derivatives, Term structure of the interest rates, bonds, exotic options, swaps, ……

- Tools: statistics, signal analysis, path (Wiener) integrals, Montecarlo simulations,
- PDEs, …

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Thank you!
Merci!

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