Introduction to Radiochemistry

Lecture 5

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The Nuclear Shell Model

Introduction

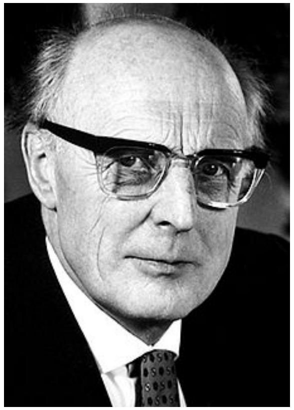
- The introduction of the **Liquid Drop Model** (a collective model) permitted the investigation of many properties of the nuclear chart.

- Still, the presence of particularly bound nuclei ("magic numbers") is unexplained.

- Need of more refined models: microscopic models.

- First one developed: **The Nuclear Shell Model**.

- 1963 Nobel to M. Goeppert-Mayer and J.H. Jensen.

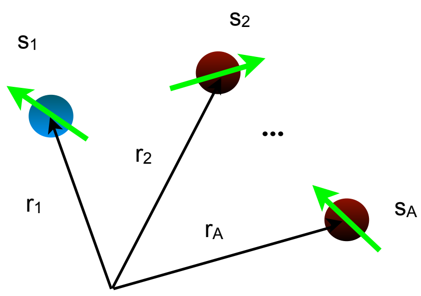


The Quantum Many-Body Problem

- The nucleus is a quantum mechanical many-body system consisting of A nucleons interacting with each other with a complex force.
  If we wanted to solve the nucleus
  we should solve the Schrödinger equation for an A-body system:

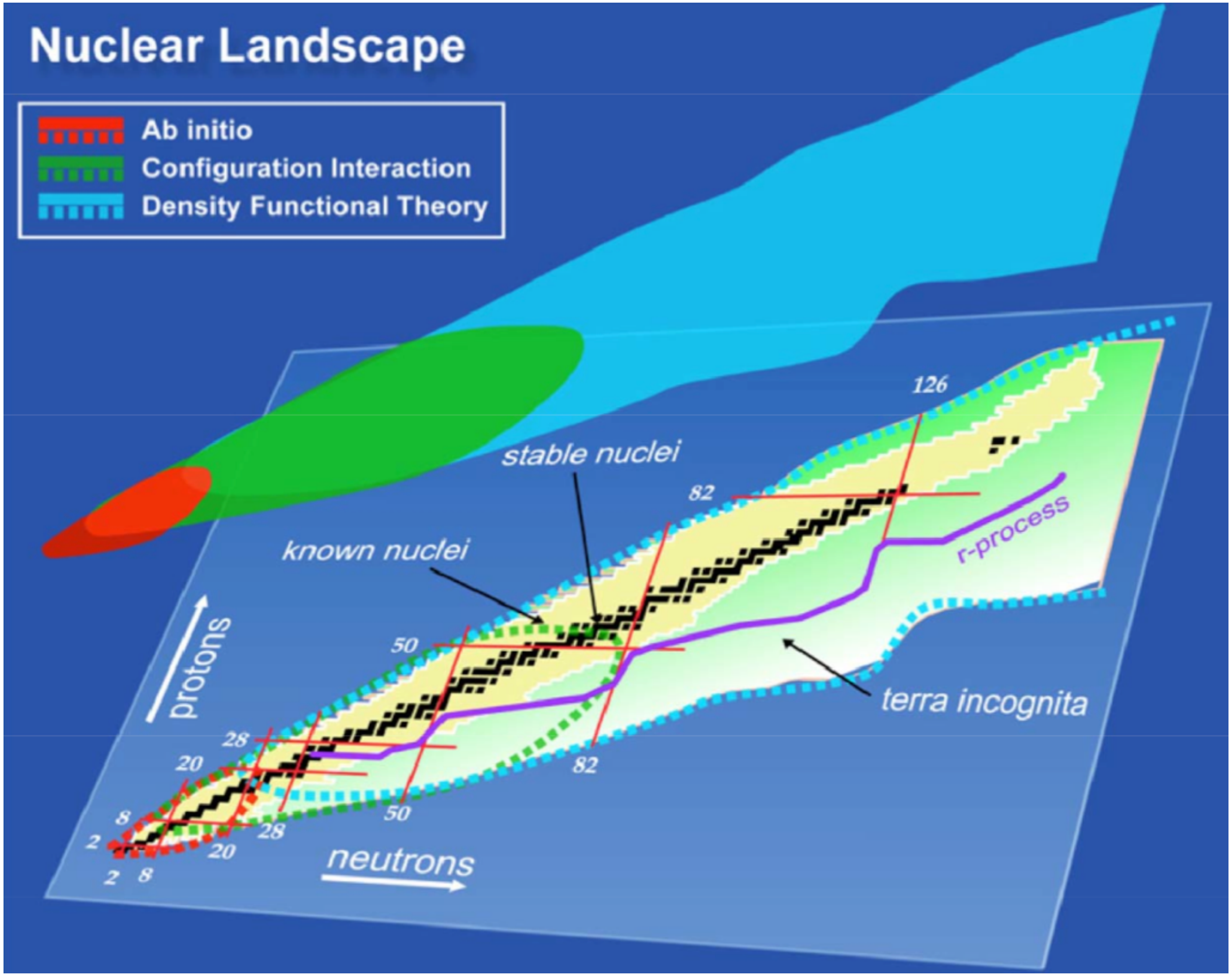
$$H|\Psi\rangle = E|\Psi\rangle$$

- The complexity of the calculation will diverge, because we have to deal with many degrees of freedom (too many coupled differential equations to solve ...)

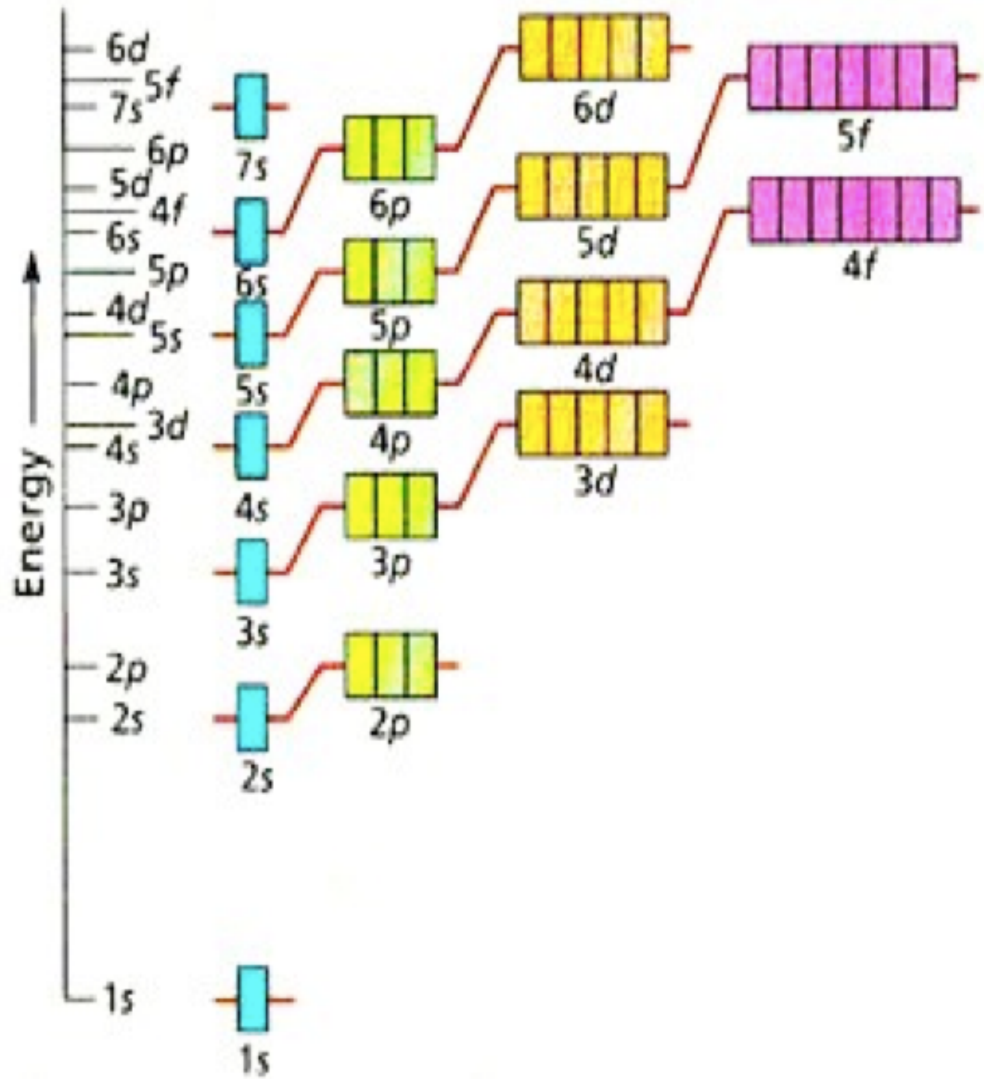


- Solving the nucleus as a many-body problem is a highly complicated task and it can be done with different degrees of approximations according to the mass number A:
after A>20-40 you have to abandon the exact microscopic theory and chose to work with an simplified theory, which should be mathematically tractable and still rich in physical insight.

The Quantum Many-Body Problem



The Shell Model in Atoms



Why a Shell Model of the Nucleus ?

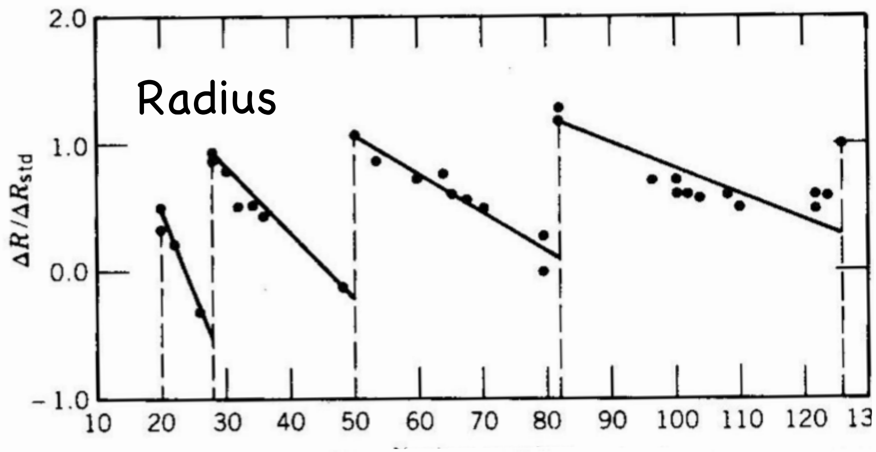
Shell structure in the nucleus would mean that individual nucleons inhabit orbitals of well defined energy. Not evident a priory why this should be the case.

Why?

- The liquid drop model is very successful in describing the binging energy. Liquid drops have a smooth behaviour with increasing size, do not exhibit jumps as matter is increased.
- No obvious centre for nucleons to orbit around.
- No external potential in nuclei, that should be the equivalent of the Coulomb force in atoms.
- The strongly repulsive nucleon nucleon potential should scatter nucleons far out of their orbits, does not seem obvious that there can be an orbit with defined energy where the nucleon sits.

But the experimental evidence seems to say otherwise!

Experimental Evidence



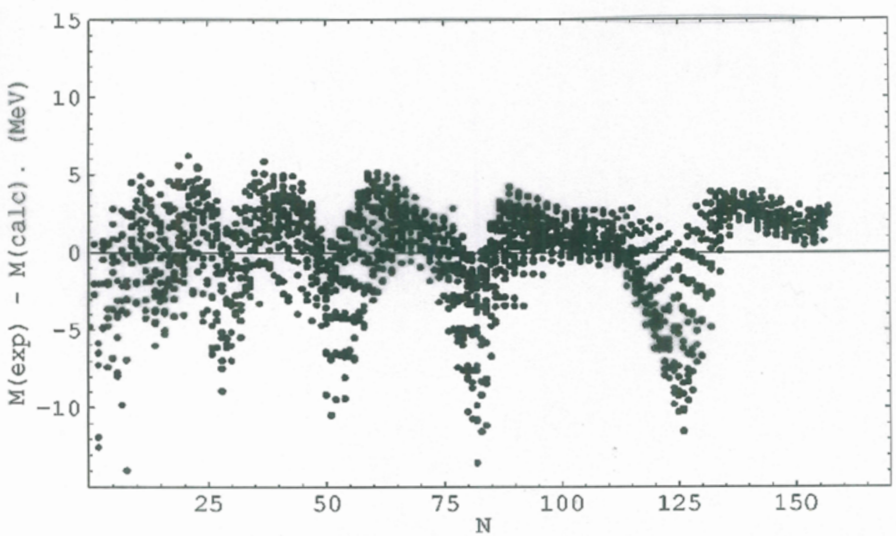
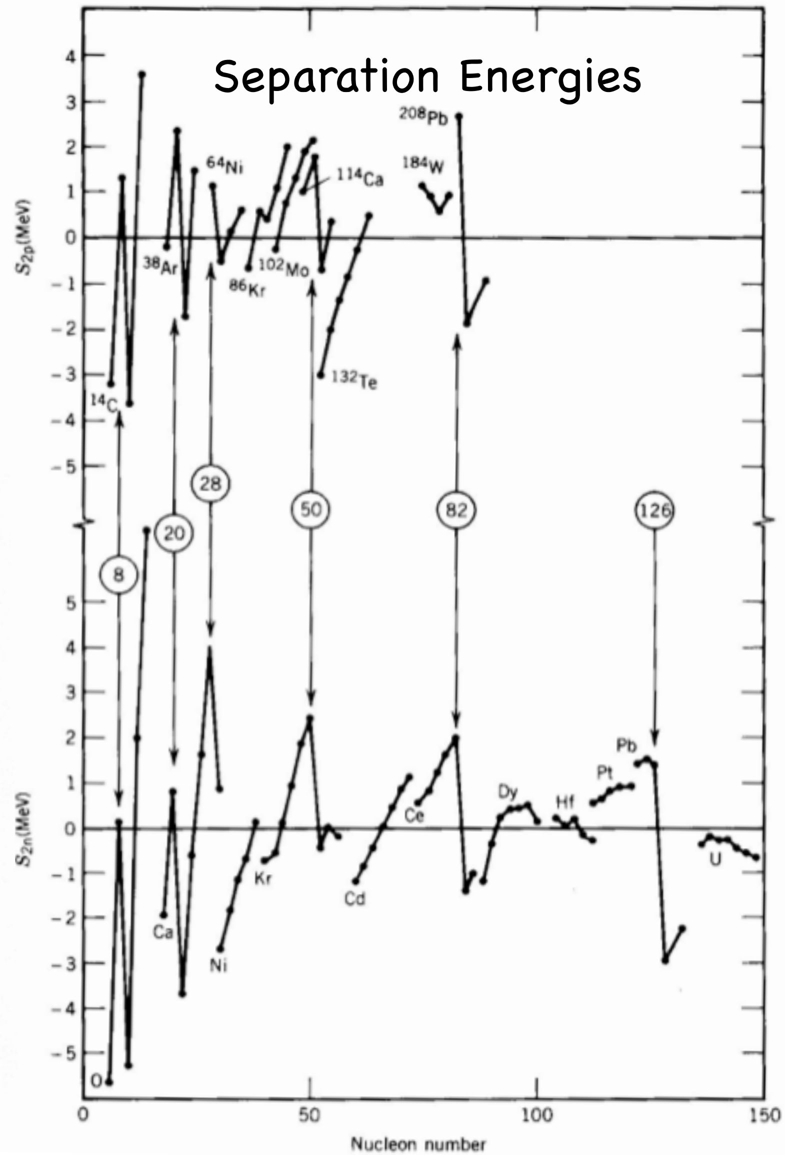


FIG. 8. Deviations from experiment of the von Weizsäcker mass formula (9), shown as a function of neutron number N.



The Nuclear Shell Model

Nuclei exhibit a shell structure:
Experimental data indicate local maxima of the binding energy and local minima of radii in proximity of the neutron or proton:

"magic numbers" 2, 8, 20, 28, 50, 82, 126. (126 only for neutrons)

The theory that explains this is called non interacting shell model or **nuclear shell model**. It is a simplified theory that accounts though for measured properties and can predict others. It is based on the assumption that the motion of the single nucleon is governed by a potential caused by all other nucleons.

Independent Particles Approximation

In an independent particle model it is assumed that particles do not interact with each other. They are only subject to the Pauli principle.

Formally this means that one can write the A-body Hamiltonian as

$$H = \sum_{i}^{A} h_i, \quad h_i : \text{single particle Hamiltonian}$$

Note: there is nothing that connects particle i with particle j

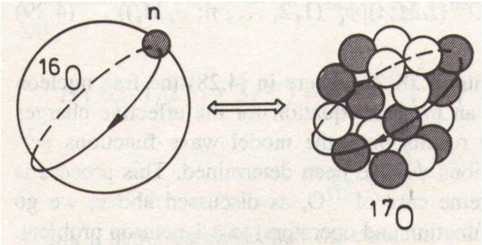
Examples:

$$h_i = \frac{p_i^2}{2m}$$ only kinetic energy (Fermi gas models with SD of plane waves)

$$h_i = \frac{p_i^2}{2m} + U_i$$

U_i is the potential felt by particle i, which could be an external potential like the Coulomb force in atoms or an average potential given to i by the presence of all the other A-1 particles.

Assumption: The interaction of a nucleon with ALL the other particles is approximated by a "mean" potential



Nuclear Shell Model

Case of the spherical HO potential

$$U_i = \frac{1}{2} m\omega^2 r_i^2 \Longrightarrow h_i = \frac{p_i^2}{2m} + \frac{1}{2} m\omega^2 r_i^2$$

$$h_i \ \varphi(\vec{r}_i) = \varepsilon_k \ \varphi(\vec{r}_i)$$ HO in 3 dimensions

k group of quantum numbers $k = n\ell m$

For every particle (omit i index) $\varphi_{n\ell m}(\vec{r}\,) = R_{n\ell}(r) \ Y_{\ell m}(\hat{r})$

analytical solution of the radial equation

spherical harmonics

n radial quantum number

ℓ, m quantum numbers related to angular momentum and its projection

$$\varepsilon_{n\ell} = \left(N + \frac{3}{2}\right)\hbar\omega = \left(2(n-1) + \ell + \frac{3}{2}\right)\hbar\omega$$

does not depend on m

$$\boxed{N = 2(n-1) + \ell + \frac{3}{2}}$$ Total oscillator quanta excited

Nuclear Shell Model

Case of the spherical HO potential

$$N = 2(n-1) + \ell + \frac{3}{2}$$ defines the shell

$n-1$ number of nodes in $R_{n\ell}(r)$

For fixed N, the angular momentum can vary because n can vary

$$\ell = N - 2(n-1) \implies \ell = N, N-2, \ldots, 2, 0.$$

N	0	1	2	...
ℓ	0	1	0 or 2	...

Because the energy does not depend on the projection m, for a given N, and ℓ we have a degeneracy

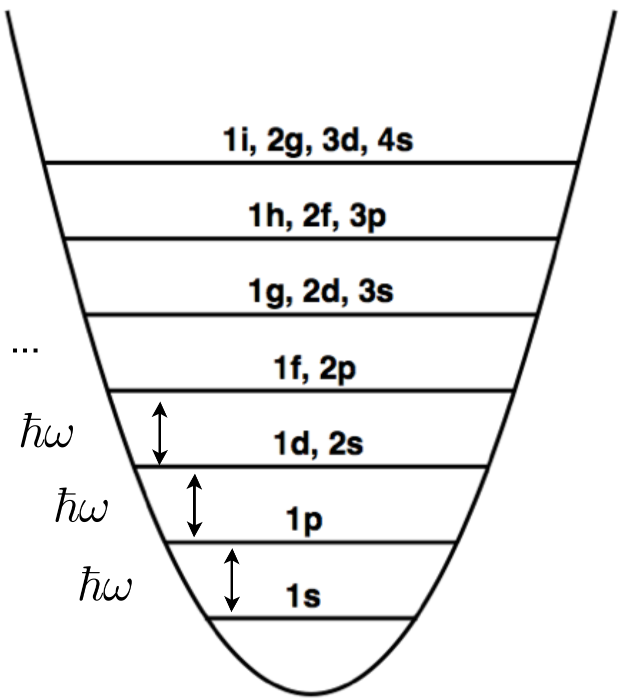
$$d_N = 2 \quad (2\ell + 1)$$

two possible projections of spin +1/2 or -1/2

all possible values of m for a given ℓ

Nuclear Shell Model

Case of the spherical HO potential



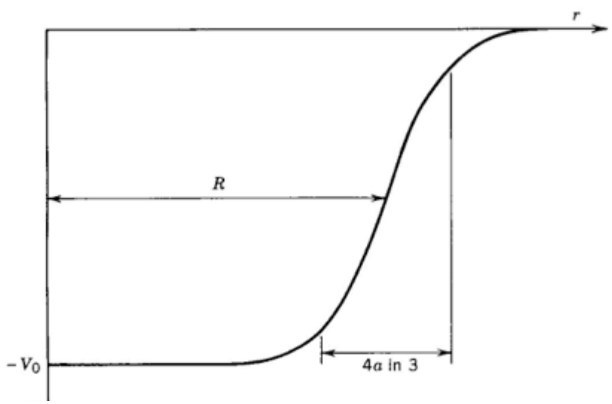
The integrated degeneracy is related to the magic numbers

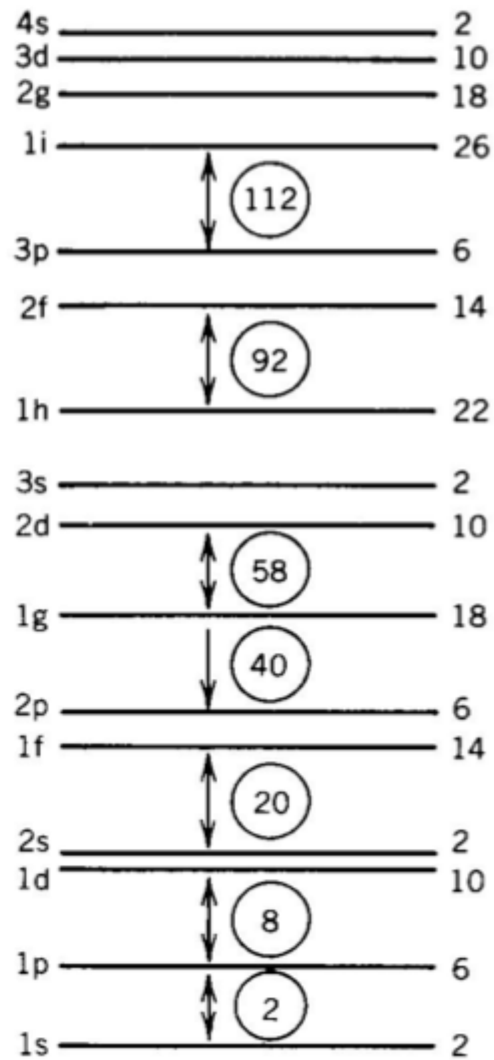
N	E_N	d_N	$\boxed{\sum_N d_N}$	$n(l)$	parity
0	$\frac{3}{2}\hbar\omega$	2	2	$1s$	+
1	$\frac{5}{2}\hbar\omega$	6	8	$1p$	-
2	$\frac{7}{2}\hbar\omega$	12	20	$1d, 2s$	+
3	$\frac{9}{2}\hbar\omega$	20	40	$1f, 2p$	-
4	$\frac{11}{2}\hbar\omega$	30	70	$1g, 2d, 3s$	+
5	$\frac{13}{2}\hbar\omega$	42	112	$1h, 2f, 3p$	-
6	$\frac{15}{2}\hbar\omega$	56	168	$1i, 2g, 3d, 4s$	+

The magic numbers are wrong after the first three!

Nuclear Shell Model

One can try to use a Wood-Saxton form for U_i





Still the empirical magic number are not reproduced
2, 8, 20,28, 50, 82, 126

None of these single particle potentials seemed to work properly...

Nuclear Shell Model

Fermi's suggestion: any evidence for a spin-orbit force?

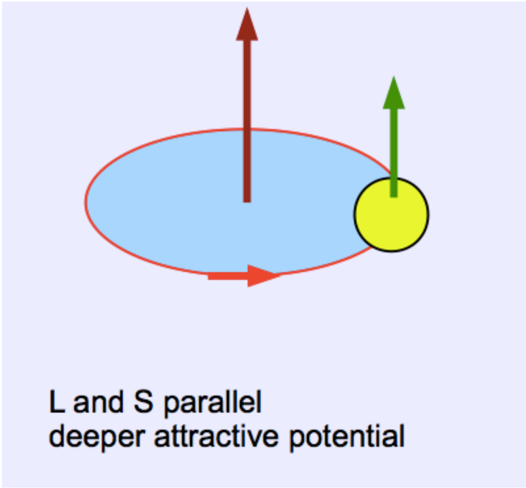
Mean field central potential plus an empirical spin-orbit term like

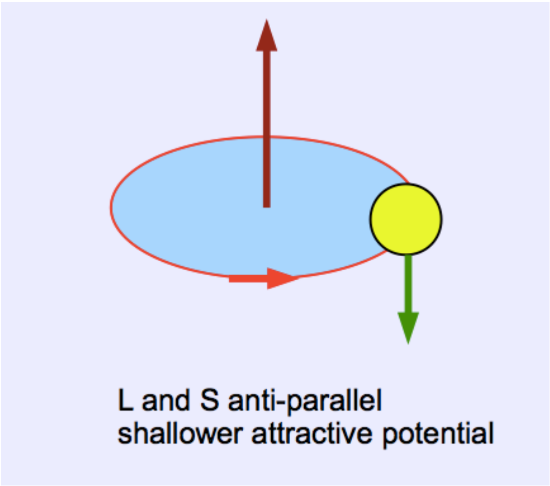
$$U(r) = V_0(r) + V_{\ell s}(r)\ \vec{\ell} \cdot \vec{s}$$

$\vec{\ell}$ orbital angular momentum

$\vec{s}$ spin (intrinsic) angular momentum

with $V_0(r), V_{\ell s}(r)$ being negative (attractive potentials)

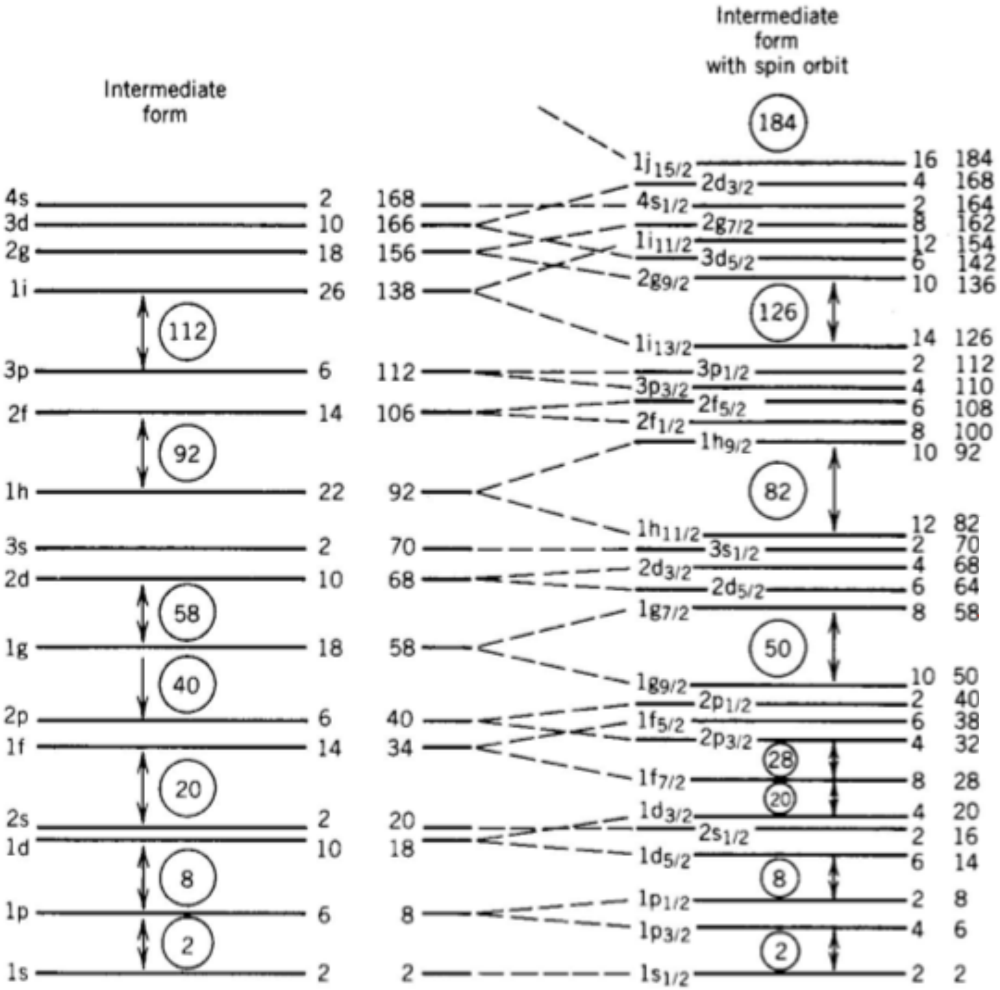




Nuclear Shell Model

With the addition of the spin-orbit, the magic numbers are reproduced

2, 8, 20,28, 50, 82, 126



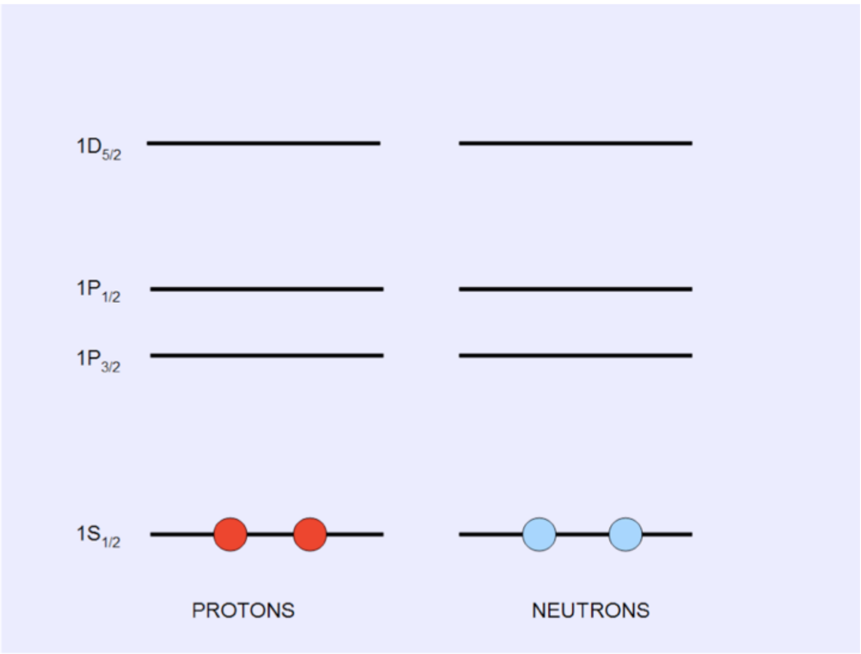
Maria Goppert-Mayer and Hans Jensen
Nobel prize in 1963

Phys. Rev. **75**, 1969 (1949)

Nuclear Shell Model

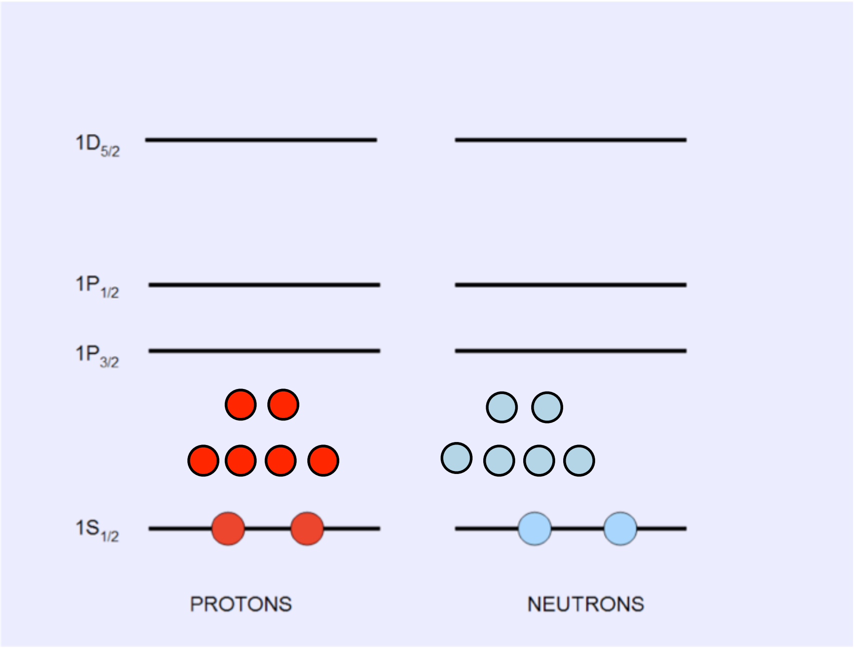
We can build the shell structure of nuclei, just like we fill up the electron shells in atoms. Only now there are separate proton and neutron shells.

^{4}He



Double closed s-shell

^{16}O

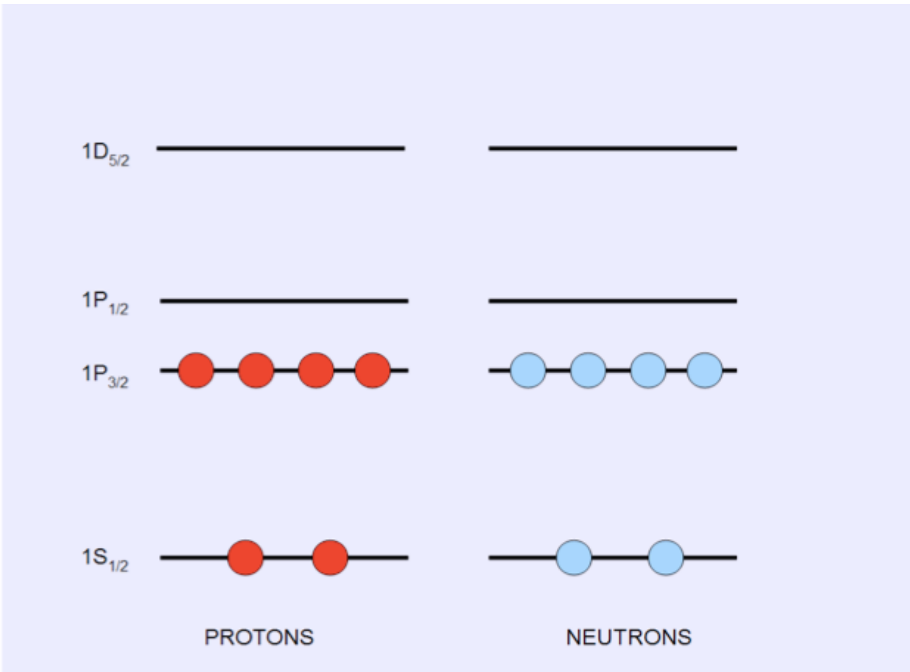


Double closed p-shell

Double magic nuclei:
extra binding energy, extra small radius, extra low reaction probability

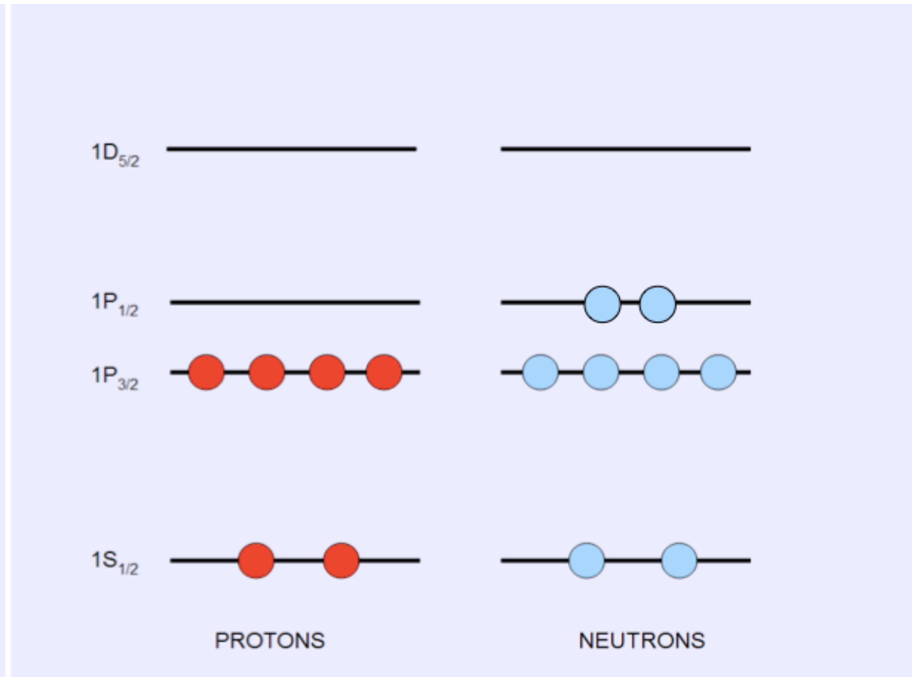
Nuclear Shell Model

^{12}C



Double closed sub-shell
$p_{3/2}$ is full (6 p + 6 n)

^{14}C



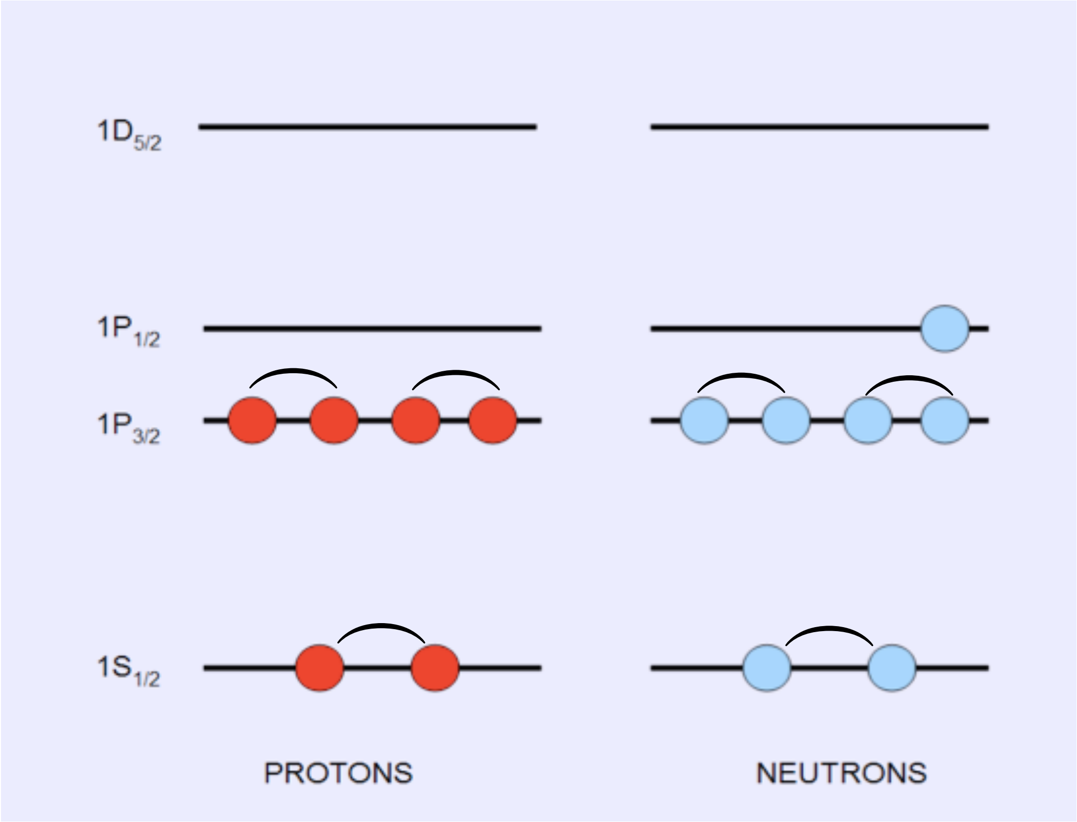
Closed p-shell for neutrons (8)
Closed $p_{3/2}$ sub-shell for protons (6)

Protons and neutrons pair off, so if there is an even number of p and n, then the total angular momentum of the nucleus in the g.s. is $J=0^+$

Nuclear Shell Model

If there is an unpaired p or n, then the total angular momentum J^p of the nucleus in the g.s. is equal to the angular momentum and parity of the single nucleon in the outer shell

^{13}C

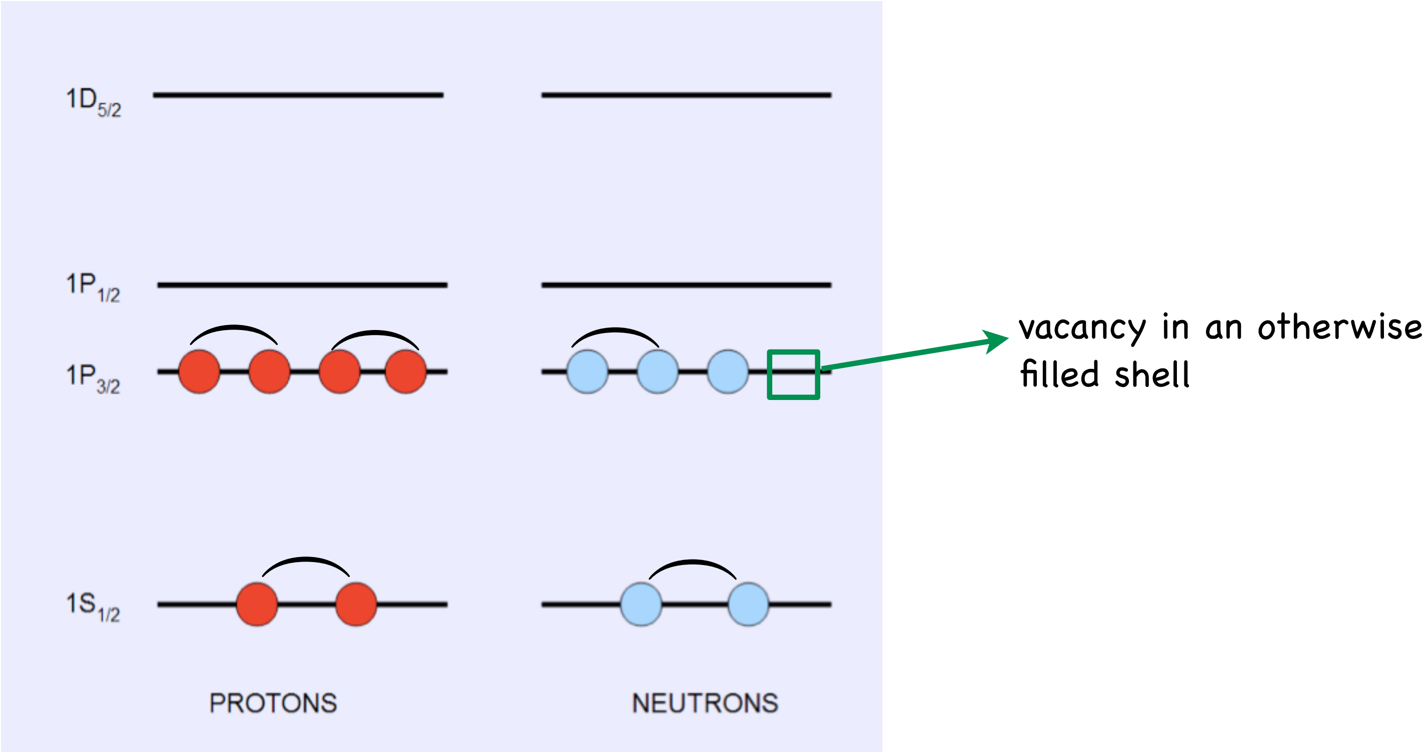


6 protons and 7 neutrons. One n in the outer p1/2 shell
The nucleus has $J=1/2^-$

Nuclear Shell Model

If there is an unpaired p or n, then the total angular momentum J^p of the nucleus in the g.s. is equal to the angular momentum and parity of the single nucleon in the outer shell

^{11}C



6 protons and 5 neutrons. One unpaired n in the p3/2 shell, the nucleus has $J=3/2^-$

A vacancy in an otherwise filled shell acts like a lone particle in that same shell in determining the spin parity of the nucleus. Thus, again, $J=3/2^-$

Summary

- Simple microscopic model: independent particles

- Different potential models: harmonic, Wood-Saxon, ...

- Nuclear Shell Model

- The Spin-Orbit Coupling

- Nuclear Shells and Magic Numbers