

Introduction to Nuclear Science

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Chapter 1

Introduction and Historic Remarks

1.1 Introduction

The field of radiochemistry started with the discovery of natural radioactivity about one century ago. The field progressed at a fast pace producing profound results in pure and applied science, together with many Noble Prizes in Physics and Chemistry. **Radiochemistry** is the application of nuclear phenomena like radioactive decays to chemical problems.

1.2 A Brief History of Radiochemistry and Nuclear Science

In the following, we give a brief historical account of the scientists and discoveries which were key in founding the new fields of radiochemistry and nuclear science. This is not the main subject of these notes, so for the sake of brevity, we will only touch other very important developments, like quantum mechanics. We will try to highlight mostly the historic parts relevant to radiochemistry and refer to other books for a more complete historical account.

The Discovery of Radioactivity

The history of radiochemistry (and also of nuclear science) begins in 1896. At the beginning of that year, **Henri Becquerel (Paris, 1852-1908)**¹ made a revolutionary discovery: he has been the first scientist detecting radioactivity. What exactly radioactivity is, will be explained in more details in the next chapters while here we will stick to the historical context.

Around that year, other groundbreaking discoveries were made: Roentgen discovered X-rays (1895) and Thompson (1897) the electron. All these observations paved

¹Henri was the son of Edmond, nephew of Antonine and father of Jean: all famous scientists.

the way to the quantum revolution at the beginning of the '900s.

Becquerel was an expert in the study of fluorescence and infrared light. In particular he studied extensively the fluorescence of minerals containing Uranium. One day he wanted to expose Uranium minerals to sunlight in order to study the induced fluorescence. He worked in Paris, which is a very rainy city during that period. Without sunlight available, he put the minerals on a photographic plate and closed everything in a drawer, wrapped in light-tight paper.

Since before exposing the minerals to light, he needed a control sample for comparison purposes he decided after a while to develop the photographic plate. To his surprise, he found the photographic plate blackened. He understood that with or without fluorescence, minerals containing Uranium were able to blacken photographic plates. He postulated that Uranium was emitting a new kind of radiation. After that, he discovered also that this radiation, like the newly discovered X-rays was able to ionize gases and induce currents in them. This observation opened another field: radiation detection and measurement.

Becquerel was also convinced that the new radiation was a property of Uranium itself and it was not connected to its chemical state (the way it was bound to other substances in the mineral).

Becquerel continued to study radioactivity but the next fundamental advances were due to other two scientists: **Maria Skłodowska (Warsaw, 1867-1934)**, better known as **Madame Marie Curie**, and her husband, **Pierre Curie (Paris, 1859-1906)**. Marie Curie started her experiments trying to reproduce Becquerel's results and then she started looking for other substances with the same properties.

She found out that some minerals containing Uranium were much more radioactive than the amount of Uranium could explain (given previous results on different minerals and after chemical analysis). Her conclusion was that some minerals contained new radioactive atomic species. Marie Curie started chemical analyses on the minerals for discovering from where the radioactivity was coming from, officially starting the field of **radiochemistry**.

In 1898, after two years of work, she isolated the new substance. She gave the new element the patriotic name of *Polonium*.

After the discovery of Polonium (belonging to the II analytic group), they found another radioactive substance in the Ca/Ba group. It took a while to separate it from Barium, but they finally succeeded (end of 1898) with fractional crystallization techniques. The new element was called Radium. It was also soon noticed that radioactivity in minerals was decaying **exponentially with time**. At that time, the Curies received help from the prominent geologist H. Suess from Vienna. He managed to ship to Paris a ton of pitchblende² minerals from the Joachimstal mine in Czechoslovakia (at that time belonging to the Austrohungaric Empire). The Curies managed to refine enough Radium for measuring its atomic weight and observed its spectral lines. In 1903, Becquerel, Pierre and Marie Curie shared the Nobel Prize in Physics for the discovery of radioactivity.

In 1911, after the death of Pierre, Marie Curie was awarded a second Nobel Prize

²The pitchblende is mainly composed by UO_2 and U_3O_8 . The ore contains also lead oxides thorium, rare earth elements and traces of helium.



Figure 1.1: Left: Henri Becquerel, Center: Marie Curie, Right: Pierre Curie.

(this time in Chemistry) for the discovery of Polonium and Radium. Pierre and Marie had a daughter: Irene (1897-1956). Irene married Frederic Joliot and together they will further contribute to nuclear science, winning a Nobel Prize for Chemistry themselves in 1935 for the discovery of artificial radioactivity.

The Birth of Nuclear Science

One of the most prominent scientists at the time of the discovery of radioactivity was **Ernest Rutherford (1871-1937)**. He was born in New Zealand and went to England to work with J.J. Thompson (at the Cavendish Laboratory, Cambridge), starting with studies on the ionization of gases induced by X-rays. After that, he quickly moved on ionization induced by the newly discovered radiation from Uranium. Rutherford discovered two different kinds of radiations emitted from Uranium and he called them α and β . After two years it was clear that the β radiation was made of electrons and P. Villard in France discovered a third kind of radiation, called, following the nomenclature of Rutherford, γ . In 1898, Rutherford won a professorship at McGill University in Montreal (Canada) where he found very favorable conditions for continuing his work. Together with the chemist **Frederick Soddy (1877-1956)**, he realized that chemical elements can transform into other elements. At that time, this hypothesis was very hard to believe (nobody believed in alchemy anymore!) and it required a lot of care from Rutherford in communicating it to the scientific community.

Rutherford noticed that α particles were able to induce scintillation not only on a fluorescence screen, but also in places far from it. His conclusion was that α particles were interacting with air, which mainly contains nitrogen, following the reaction



After the Canadian experience, Rutherford accepted a professorship in Manchester and went back to England. In Manchester, he finally proved that the α radiation was made of ionized helium and in 1908 he was awarded the Nobel Prize for Chemistry. He was helped in his research by **Hans Geiger (1882-1945)**, the inventor of the well-known radiation detector called after his name. In 1909, **Ernest Marsden (1889-1970)** came from New Zealand to England for working with Rutherford. He studied the behavior of α radiation in materials, observing that sometimes they were



Figure 1.2: Left: Ernest Rutherford, Center: Frederick Soddy, Right: Hans Geiger.

deflected at very high angles. In 1911, Rutherford understood what was happening in the experiments of Marsden: the α particles were scattering against a dense object inside the atoms while the electrons were orbiting around it. Rutherford called the object **nucleus**. With this planetary model of the atom, nuclear science was born. In 1913, **Antonius van der Broek (1870-1926)** demonstrated that the elements in the periodic table should be ordered according to the atomic number (related to the charge of the nucleus) and not according to the atomic weight, as Mendeleev did in his ingenious idea in 1869.

In 1912 J.J. Thompson found that the existence of isotopes (a word invented by Soddy) was not limited to radioactive substances, but it was a general feature of all the atomic elements. The fact that atomic number and atomic weight were in general different lead Rutherford to postulate the existence of a new particle inside the nucleus.

This elusive particle, finally discovered by **James Chadwick (1891-1974)**, in 1932 was called **neutron**. The first rudimentary picture of the structure of matter was finally complete: atoms were made by a nucleus and orbiting electrons. The nucleus was made by two particles similar in weight: protons (positively charged) and neutrons (electrically neutral).

More Nuclei and new (Anti)Particles

1932 was a memorable year for scientific discoveries. After the identification of the neutron by Chadwick (1935 Nobel Prize for Physics), it was the turn of deuterium (the first discovered hydrogen isotope) and the positron (the first discovered antiparticle). Deuterium was first clearly isolated by fractional distillation of liquid hydrogen by **Harold Urey (1893-1981)**. He proved its existence by spectroscopy. This discovery explained why the atomic weight of hydrogen was slightly different from 1. Deuterium is present also in ordinary water in 1 part over 6000. Later, Urey contributed to the development of nuclear energy and weapons inventing an isotopic separation method for Uranium enrichment based on gaseous diffusion.

Another groundbreaking discovery came from the study of cosmic rays, which were known since the beginning of the 900s. It was only after the work of **Victor Hess (1883-1964)**, who investigated them via balloon experiments, that their extraterrestrial origin was demonstrated. **Carl Anderson (1905-1991)** at the California Institute of Technology was the first investigating cosmic rays with detectors (“Wil-



Figure 1.3: Left: James Chadwick, Center: Harold Urey, Right: Carl Anderson.

son Chambers”) immersed in a magnetic field. With this technique, he was able to detect particles behaving like electrons, but with opposite charge.

Hess and Anderson shared the Nobel in Physics 1936 for the discovery of the first antimatter particle: the positron. A remarkable fact is that such particle was already predicted on purely theoretical ground by **Paul Dirac (1902-1984)**, one of the founders of quantum mechanics.

Another important discovery happened in 1934, when the Joliot-Curies (Irene, Maries’ daughter and Frederick Joliot) announced the discovery of **artificial radioactivity**. Previously they observed both the positron and the neutron, without recognizing them. Their disappointment was compensated with the Nobel Prize in Chemistry for the discovery of artificial radioactivity. The Joliot-Curies observed that when an Aluminum foil was irradiated by α particles from Polonium, it emitted positrons. Even when the Polonium was removed, Aluminum kept emitting positrons with an exponentially decreasing intensity. The reaction observed was



where the top index is the mass number and the bottom one the atomic number. With reaction 1.2, they created artificially the radioactive nucleus ${}_{15}^{30}\text{P}$ which kept decaying after the removal of Polonium:



with a half life of about 2.5 minutes.

Accelerators

After Rutherford’s experiments, it was clear the need of a controlled source of particles for nuclear experiments. At the beginning, only radioactive elements or cosmic rays were available, but scientists started thinking about an artificial source which has to be more intense, controlled and (most importantly) more energetic. After many failed attempts, the first successful particle accelerator was designed and constructed by **John Cockcroft (1897-1967)** and **Ernest Walton (1903-1995)** (both awarded the Nobel Prize in Physics in 1951) at the Cavendish Laboratory in England. They accelerated lithium ions at energies of 700keV and observed the first

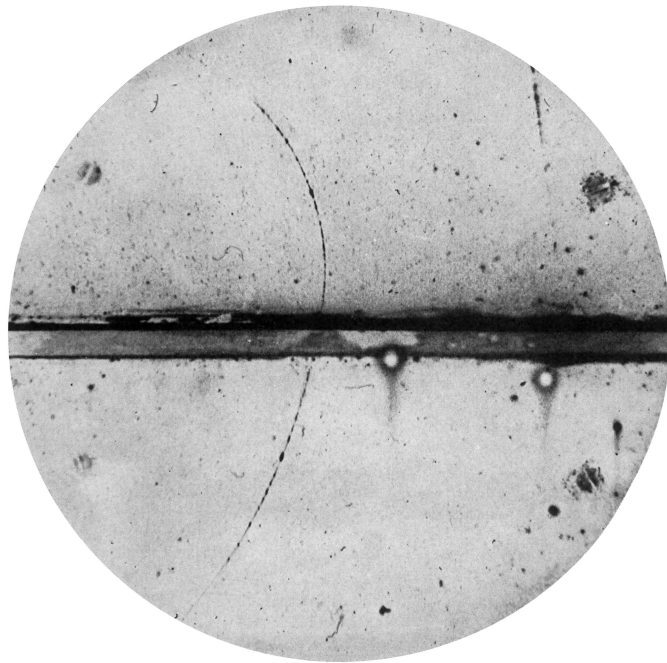


Figure 1.4: This picture comes from one of Anderson's experiments (Phys.Rev. 43, 491, 1932) with a Wilson chamber immersed in a magnetic field. The positron is coming from below and its trajectory is curved because of the magnetic field. In the middle, there is a lead plate where the positron loses energy: indeed the positron track is more curved after it.

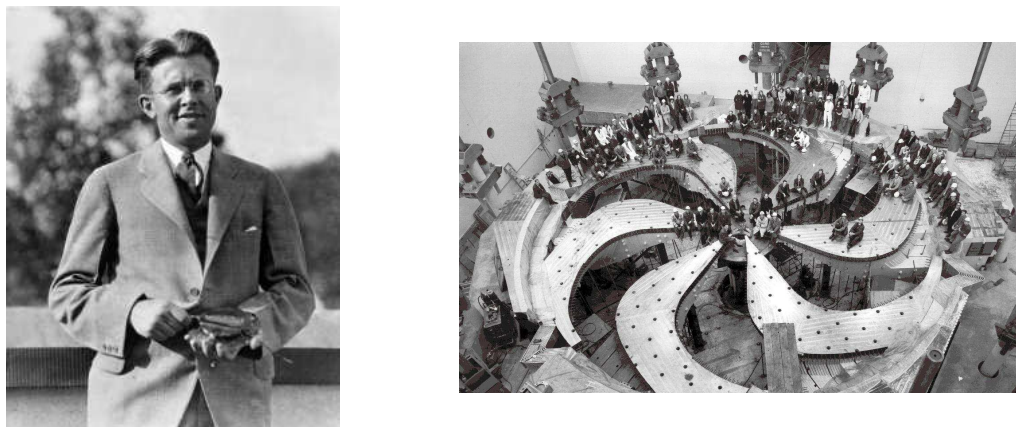


Figure 1.5: The smallest and the largest cyclotrons. Left: Lawrence holds in his hands a tiny cyclotron prototype. Right: the TRIUMF cyclotron in Vancouver, BC (Canada) during its construction in 1972. It is still the largest cyclotron of the world.

induced nuclear disintegrations.

Further progress in rising the energy of ions was possible only after the introduction of the idea of multiple (or staged) acceleration. The first pioneer in this field was **Ernest Lawrence (1901-1958)**, who introduced and realized the idea of the cyclotron. He realized bigger and bigger versions of this machine, tirelessly looking for funding. He neglected nuclear research and devoted all his energies uniquely at the development of the machine and used it mostly for medical applications which was a better source of funding. The capabilities of the new machine were immense with respect to the small-laboratory based enriching techniques of the time. With the cyclotron, it was possible to induce radioactivities orders of magnitude higher than obtained before. After the cyclotron, many other kinds of accelerators were developed and nowadays energies in the TeV range have been reached. The cyclotron, after many decades after its invention, still remains one of the best tools available to the nuclear scientist for research, medical and material science applications.

Enrico Fermi, Nuclear Energy and Elements beyond Uranium

Enrico Fermi (1901-1954) is one of the greatest physicists of history and he is remembered for many contributions to both theoretical and experimental physics. After the discovery of the Joliot-Curies, he decided to use neutrons instead of α particles for inducing radioactivity. The idea was simple in principle: since the α particle is positively charged as the target nuclei, they will repel each other. This was the reason why the first experiments had poor efficiency: every 10^6 α particles, an Aluminum nucleus was affected (see Eq. 1.2). Neutrons, being electrically neutral should have a higher probability to induce nuclear reactions. Fermi and collaborators started bombarding substances systematically with growing atomic number. The first results came with Fluor. At the end of 1934, Fermi started irradiating Uranium finding results difficult to interpret. At a certain point he believed to have

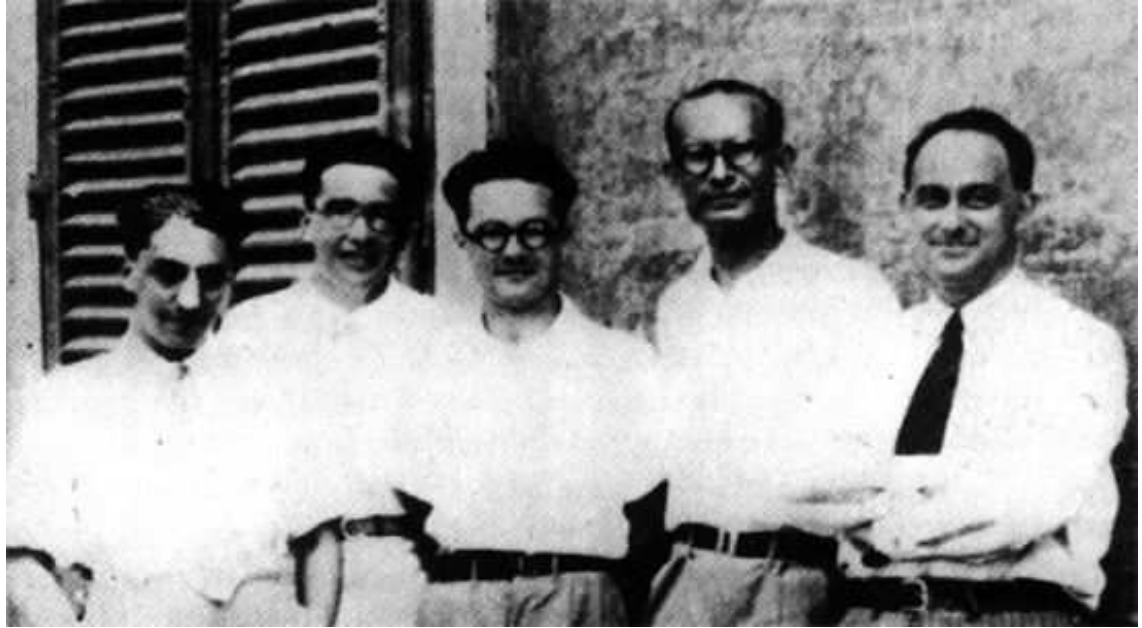


Figure 1.6: From left: O. D'Agostino, E. Segre, E. Amaldi, F. Rasetti and E. Fermi. Together with B. Pontecorvo (not in this picture), they authored the work on slow neutrons. They are known as “the boys from via Panisperna”, from Panisperna Street in Rome, where the Physics Institute was at that time.

created new artificial elements. In the same year, he found out that neutrons were significantly slowed down if filtered through paraffin. Paraffin contains a lot of hydrogen: neutrons were losing energy hitting protons (hydrogen nuclei). The finding was interesting by itself, but the application was more astonishing: slow neutrons were much more effective in inducing artificial radioactivity. In 1938, Fermi won the Nobel Prize for his research with neutrons and since the political situation in Italy was deteriorating, from Stockholm he traveled directly to USA with his family, where he remained. When Fermi arrived in New York, **Niels Bohr** (another father of quantum mechanics) announced that while he was receiving his Nobel Prize, **Otto Hahn (1879-1968)** and **Friedrich Strassmann (1902-1980)** in Berlin discovered the fission of Uranium. Hahn has been a student of Rutherford in Canada. Important contributions were previously given also by **Lise Meitner (1878-1968)**. At that time she was already removed from her working place because of the nazist racial laws, finding a safer place in Sweden.

The discoveries of nuclear fission and slow neutrons paved the way to nuclear energy with its pacific and military applications.

In 1940, McMillan and P. Abelson discovered the first transuranic element ($Z=93$) and called it Neptunium (^{239}Np). In the same year, **Glenn Seaborg (1912-1999)**, **Edwin McMillan (1907-1991)** and collaborators discovered Plutonium (^{238}Pu) irradiating Uranium with deuterons delivered with the cyclotron at University of California, Berkeley.

In 1942, Fermi and collaborators realized the first controlled chain reaction with

1.2. A BRIEF HISTORY OF RADIOCHEMISTRY AND NUCLEAR SCIENCES



Figure 1.7: Left: Otto Hahn, Center: Friedrich Strassmann, Right: Lise Meitner.

Uranium.

After the second World War, many more transuranic elements were created. In the spirit of cold War, USA and URSS scientists started a race to high-Z elements, reaching up to $Z=105$. The heaviest element to date has $Z=118$. It is currently named Ununoctium (Uuo) and has still to be approved by IUPAC (International Union of Pure and Applied Chemistry). The highest-Z element approved by IUPAC is currently the Livermorium (Lv) with $Z=116$. The name stems from the laboratory where it was discovered: the Lawrence Livermore National Laboratory (California, USA).

Chapter 2

General Concepts

2.1 Notation, Units, and Useful Formulas

Atomic nuclei are very small objects and a typical unit of measure used in the field is the Fermi (fm). $1\text{fm} = 10^{-15}\text{m}$. The typical size of a nucleus is of the order of 1fm. In the following, we will encounter the concept of cross section, which is related to the concept of surface. Nuclear cross sections are commonly expressed in barns (b). $1\text{b} = 10^{-28}\text{m}^2$.

The mass of a proton is $m_p = 1.6 \times 10^{-27}\text{Kg}$ while the neutron is about 0.14% heavier. A convenient mass unit is the **atomic mass unit** (u) and $1\text{u} = 1.6605402 \times 10^{-27}\text{Kg}$. The atomic mass unit is defined using as reference the neutral ^{12}C atom:

$$u = \frac{m(^{12}\text{C})}{12} = \frac{1\text{Kg}}{N_A} = 1.6605402(10) \times 10^{-27}\text{Kg} = 931.49432(28)\text{MeV}/c^2. \quad (2.1)$$

$N_A = 6.0221367(36) \times 10^{26} (\text{kg mol})^{-1}$ is the Avogadro number. The numbers in parentheses indicate the uncertainty in the last digits.

MeV means millions of eV (electronvolts): one eV is the energy acquired by a particle of with unit electric charge if accelerated with a potential of 1 Volt. A unit of electric charge is equivalent to 1.6×10^{-19} Coulombs. Summarizing the energy unit conversions:

$$1\text{eV} = 1.602 \times 10^{-19}\text{J} = 1.60219 \times 10^{-12}\text{erg} \quad (2.2)$$

From the above definitions, the mass of ^{12}C is exactly 12u, while for the proton and the neutron:

$$\begin{aligned} M_p &= 1.007276470(12)\text{u} \\ M_n &= 1.008664898(12)\text{u} \end{aligned}$$

2.2 Notation

Nuclides

A nucleus is uniquely identified by its atomic number Z (which corresponds to the number of protons) and the mass number A (which corresponds to the total number

of nucleons). The number of neutrons N is easily obtained: $N=A-Z$. Nuclei with same Z but different A (and therefore different N) are called **isotopes**. Nuclei with same N but different A (and therefore different Z) are called **isotones**. In nuclear science in general, and in radiochemistry in particular, a specific atomic species with nuclear values (A,Z) is called a **nuclide** and if it is radioactive, it is called a **radionuclide**. The notation identifying a specific nucleus is:

$${}^A_ZX \quad (2.3)$$

where X is a generic chemical element. For example, for tritium, the hydrogen isotope with $A=3$ we have: ${}^3_1\text{H}$.

Note that A and Z are always integer numbers since they count the number of protons or nucleons in the nucleus. The **atomic mass** refers instead to the average mass of all the isotopes of an element weighted with the proportions with which it is found in nature. The atomic mass is expressed in g/mol. For example, the atomic mass of hydrogen is 1.0079 and this informs us that the large majority of hydrogen in nature has $A=1$ with tiny addition of heavier isotopes.

Nuclear Reactions

In the previous chapter, we gave some examples of nuclear reactions, like:



A more compact notation for reactions was first proposed by **Walther Bothe (1891-1957)**.

The last equation can be conveniently written as:

$${}^{27}_{13}\text{Al}(\alpha, n){}^{30}_{15}\text{P} \quad (2.5)$$

The meaning of the last expression is:

- The leftmost element is the target
- The first particle in the parenthesis is the incident one
- The lighter emerging particle(s) is (are) in the second part of the parenthesis.
- The rightmost element is the heavier produced particle (or nucleus).

2.3 Binding Energy and Q-value

The **binding energy** E_B of a nucleus is the amount of energy needed for removing all the protons and neutrons from it. It is given by the mass difference between the single nucleons and the nucleus:

$$E_B(Z, N) = \{Zm_p + Nm_n - M(Z, N)\}c^2 \quad (2.6)$$

where $M(Z,N)$ is the mass of the nucleus.

The **Q-value** of a reaction is the amount of energy released or absorbed by a nuclear reaction. If the Q-value is positive, the reaction produces energy while if it is negative, it requires energy for happening.

2.4 Electric Potential and Fine Structure Constant

In many calculations involving nuclei, it is useful to quickly estimate the classical electric potential between them. The electric potential between two particles with a and b elementary charges e and distance R is (mks system):

$$V = \frac{1}{4\pi\epsilon_0} \frac{(ae)(be)}{R} \quad (2.7)$$

It is convenient to introduce an adimensional number, called **fine structure constant** α :

$$\alpha = \frac{1}{4\pi\epsilon_0} \frac{e^2}{\hbar c} \approx \frac{1}{137} \quad (2.8)$$

with $\hbar = h/2\pi$. We can now rewrite the potential with the help of α :

$$V = (\alpha\hbar c) \frac{ab}{R} \approx 1.44 \frac{ab}{R} \quad (2.9)$$

In the last expression, we have $\alpha\hbar c \approx 1.44 \text{ MeV}\cdot\text{fm}$ and R is now expressed in fm. Another useful relation is $\hbar c \approx 197.3 \text{ MeV}\cdot\text{fm}$.

2.5 General Properties of Nuclei

In the following, we list some general aspects of nuclei which emerged after a century of theoretical and experimental efforts.

- Stable nuclei are found between $Z=1$ (hydrogen) and $Z=82$ (lead). These nuclei belong to the so-called **valley of stability** and this name comes from the binding energy chart as a function of Z and N (see figure).
- For every Z , there are different **isotopes** differing by the number of neutrons. The chemistry of an element is determined by the number of electrons, which is equal to Z , therefore all the isotopes have the same chemical properties.
- Unstable nuclei (outside the “valley”) decay after a certain time. Natural unstable isotopes are only the ones with half-lives comparable (or longer) to the lifetime of our solar system ($\approx 5 \times 10^9$ years). Unstable nuclei are created continuously in stars or artificially in laboratory.
- Roughly, stable nuclei have the $Z=N$ property. This approximation becomes worse as Z increases. After a certain point, the Coulomb repulsion among protons is too high and only an increasing amount of neutrons is able to keep the nucleus together. In any case, a too big N with respect to Z (or the other way around) brings the nucleus farer away from the valley of stability.

- The lighter nucleus is hydrogen with just one proton. The next hydrogen isotope is called Deuterium and it has one proton and one neutron. Nuclei with only two protons or two neutrons are not bound: this means that the n-p force is in general attractive but it is not the case for p-p or n-n.
- The rich variety of nuclei and their properties have roots in the nuclear force.

Chapter 3

Radioactive Decays

3.1 Introduction

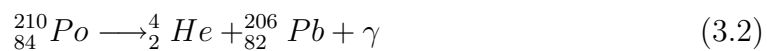
A **radioactive decay** is the spontaneous emission of one or more particles from an atomic nucleus. The fundamental properties of the nucleus (Z,N) might or might not change in the process. In general, the rate of decay is dictated by nuclear dynamics and it has no relation to factors influencing chemical reactions, like temperature, pressure, physical state or reaction speed. There are three possible radioactive decays, named α , β and γ , following a notation introduced by Rutherford.

3.2 Alpha Decay

The α decay consists in the emission of an ${}^4\text{He}$ nucleus (Z=2,A=4) from a parent nucleus M_1 . The daughter nucleus M_2 will be a lighter chemical element:



The α decay might be accompanied by the additional emission of a γ ray. Examples are



with an half-life of 138 days. The α decay is energetically favorable almost only for nuclei with $A > 150$, with few exceptions (the rare earths ${}^{144}\text{Nd}$, ${}^{147}\text{Sm}$, ${}^{148}\text{Sm}$). The lifetimes for this decay are relatively long and the energy of the emitted α tends to be in the 5-9 MeV range. The fact that the energy range is quite small but the lifetimes vary over several orders of magnitude was an important observation which eventually lead to the discovery of the quantum tunneling effect.

In fact, it is observed empirically that the α decay probability W per unit time (related to the decay time) is related to the energy of the α particle E_α as

$$\log_{10} W = a - \frac{b}{\sqrt{E_\alpha}} \quad (3.3)$$

Then coefficients a and b are weakly dependent from Z but have a stronger dependence from $N=A-Z$. The empirical law expressed by Eq. 3.3 is known as **Geiger-Nuttall** law. At this stage, it is still unclear why a nucleus finds more energetically

favorable to emit an He nucleus instead of single nucleons. For now, we just observe that nucleons prefer to cluster in 2p-2n groups inside a nucleus. A better explanation will be given when we will discuss nuclear structure models and the nuclear force.

3.2.1 Quantum Tunneling

Classically, a massive particle cannot overcome a potential barrier if it is higher than its kinetic energy. Quantum mechanically, this process is possible with a certain probability.

For understanding the tunneling process in α decay, we have to consider two competing forces: the Coulomb repulsion the α particle feels from the other positive charges in the nucleus and the nuclear force trying to keep the particle bound.

Let us consider an opposite process: an α particle coming close to a nucleus and experiencing Coulomb repulsion (both are positively charged). Let's call r the distance between the approaching α and a nucleus with Z protons. Imagining the nucleus as a uniformly charged sphere with radius R and volume proportional to the number A of nucleons in it, we can write: $R = r_0 A^{1/3}$ (see next chapter).

Outside the nucleus ($r > R$) we have a repulsive Coulomb potential

$$V(r) = \frac{1}{4\pi\epsilon_0} \frac{Ze \cdot 2e}{r} = \frac{2.88 \cdot Z}{r} \text{MeV} \quad (3.4)$$

Now we have a potential expressed in MeV with a distance r expressed in femtometers (fm) (see the previous chapter). Let's now model the attractive nuclear force with a square well as wide as R fm. The α particle is bound in the nucleus when $r < R$. The height of the potential barrier keeping the α in the nucleus can be estimated calculating the work needed to overcome the Coulomb repulsion for bringing the α at the surface of the nucleus. Using Eq. 3.4 with $R=r$, $Z=92$ and $A=238$ (^{238}U case):

$$E = \frac{1}{4\pi\epsilon_0} \frac{Ze \cdot 2e}{R} \approx 35 \text{MeV} \quad (3.5)$$

The calculated energy should be about the energy of the emitted α particle which in reality is one order of magnitude smaller, as we have seen before. Therefore, a purely classical explanation of the process does not work and we have to use quantum mechanics in order to understand the decay.

If we keep the one-dimensional square well model for the potential confining the α , we can calculate, using quantum mechanics, the probability for tunneling through it. This probability is commonly referred as transmission coefficient T and a standard textbook calculation gives

$$T = \left[1 + \frac{V^2}{4E_\alpha(V - E_\alpha)} \sinh^2 \frac{b}{\hbar} \sqrt{2m(V - E_\alpha)} \right]^{-1} \quad (3.6)$$

where m is the mass of the particle, V is the height of the potential and b the width. Since we are considering a tunneling, $E_\alpha < V$. In the special limiting case where

Element	E_α (MeV)	Half-life
^{206}Po	5.22	8.8 days
^{208}Po	5.11	2.9 years
^{210}Po	5.31	138 days
^{212}Po	8.78	0.3 μs
^{214}Po	7.68	164 μs
^{216}Po	6.78	0.15 s

Table 3.1: Half-lives of α -emitting Polonium isotopes. Note the similar α energies while the half-life varies over several orders of magnitude: a feature explainable only with quantum mechanical tunneling.

$$V \gg E_\alpha,$$

$$T \approx e^{-\frac{2b}{\hbar} \sqrt{2m(V-E_\alpha)}} \quad (3.7)$$

The transmission coefficient represents the probability of tunneling. It is already clear from the previous equation how the Geiger-Nuttall (probability proportional to $1/\sqrt{E_\alpha}$) law can emerge from a quantum calculation, thus proving the tunneling origin of the α decay¹.

The above calculation represents a very simplified model, since in reality we should consider the true shape of the potential well in three dimensions which complicates considerably the calculation of T which actually becomes not analytical although the main result still holds.

3.2.2 Recoil Energy

The mass of an α particle with respect of an atomic nucleus is significant. Therefore, after the decay, it is expected that the nucleus will acquire a recoil energy which we will now estimate.

Using the expressions for momentum ($p=mv$) and kinetic energy ($E=\frac{1}{2}mv^2$) we can derive $p^2 = 2mE$. In the following the subscript α will indicate the α particle and N the daughter nucleus after the decay.

Using momentum conservation, $p_N = p_\alpha$ and substituting $p^2 = 2mE$ after some simplifications we obtain the nucleus' recoil energy:

$$E_N = \frac{m_\alpha}{m_N} E_\alpha \quad (3.8)$$

As expected, the heavier the nucleus, the smaller its recoil.

¹Note the the Geiger-Nuttall law was originally expressed in terms of W, the probability per unit time while here we calculated a probability. The two quantities are proportional through the frequency of the α reaching the nuclear surface and the probability of an α cluster formation inside the nucleus. Both factors are not easy to calculate from first principles.

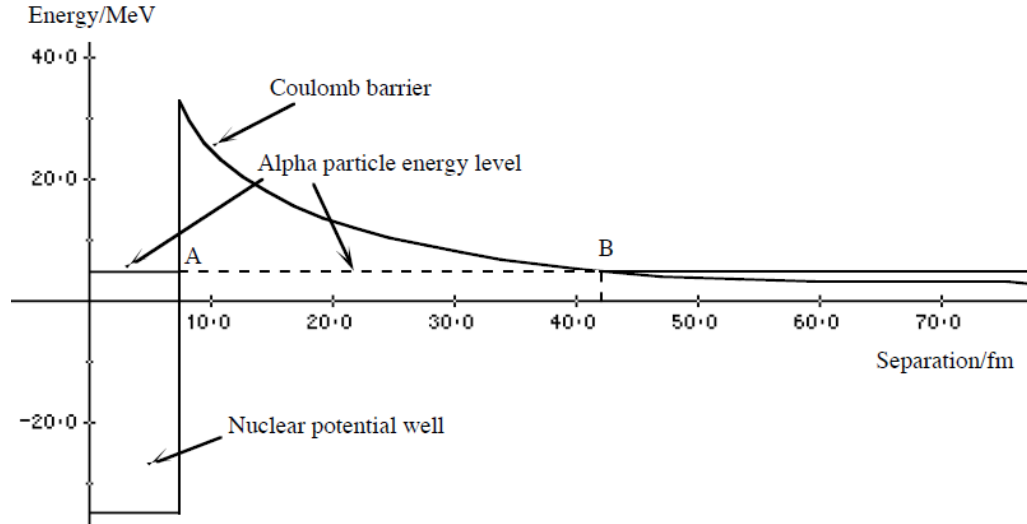


Figure 3.1: Quantum tunneling for the α decay. The nuclear binding potential is modeled as a unidimensional square well.

3.3 Beta Decay

The β decay does not change the mass number A , since there are no nucleons in the decay products, but Z and N are changed. A generic β decay of a nucleus M_1 into a daughter nucleus M_2 is represented by the following reaction:



What is happening is that one neutron decays into a proton:



The $\bar{\nu}_e$ is an electronic antineutrino. The neutron decay is a process mediated by the **weak force**². An example of β decay is



${}^{90}Sr$ is a typical β emitter used in laboratory applications and its half-life is 29.1 years. β decay happens more likely in neutron-rich nuclei, *i.e.* when the ratio N/Z is large. There is no simple relationship between energy of the emitted electron and the half-life (as for the α decay) since in this case there is a dependence from other quantum properties of the nucleus (spin, parity, etc). The α decay was a two-body

²In the Standard Model of particle physics, it is assumed that a quantum number called lepton number L is conserved. It is like charge conservation but it involves a family of particles called leptons. The electron has, say, $L=+1$. Since at the beginning there were no leptons, L was zero. In order to have $L=0$ also after the decay, the electron must be produced in association with a particle with $L=-1$: the antineutrino. Antiparticles have always the opposite quantum numbers with respect to particles.

decay and therefore the energy of the α particle and of the recoil nucleus were fixed. If we measure the energy of electrons emitted in a β decay, we will observe a wide distribution, instead of a single peak. This fact is explained observing that in this case we have a 3-body decay and the energy can be distributed in many different combinations among the three final-state particles. The energy carried away by the neutrino is usually not observed, since detecting it is a very hard task.

When the energy spectrum of the β decay was first observed, it looked like the decay was violating energy and angular momentum conservation. **Wolfgang Pauli** in 1930 postulated the existence of a new neutral and very light particle (therefore “neutrino”) for reconciling the known conservation laws. The neutrino was detected for the first time only in 1953.

The β^+ Decay

In this type of decay, a positron (e^+) is emitted instead of an electron. This means that a proton turns into a neutron and Z diminishes by one unit while A stays constant:



Note that in this case for conserving the lepton number we have a neutrino instead of an antineutrino. This decay happens likely in proton-rich nuclei, ie when N/Z is low. A β^+ decaying nucleus often used in laboratory applications is



with an half-life of 2.6 years. An interesting use of this isotope is for producing back-to-back gamma ray couples: the emitted positron annihilates with a surrounding electron and two gamma rays are produced ($e^- + e^+ \longrightarrow \gamma\gamma$)³. By momentum-energy conservation, since the mass of the electrons is $m_e=0.511$ MeV/ c^2 and the annihilation happens at rest, the energy of the γ rays will be 0.511 MeV for each.

3.4 Energetics of the β decay

In a nuclear reaction, labeling with “i” the initial reactants and with “f” the final state products, the energy conservation can be written as

$$m_i c^2 + T_i = m_f c^2 + T_f \quad (3.14)$$

where T is the kinetic energy. Given the last equation the Q value can be written as

$$Q = m_i c^2 - m_f c^2 = T_f - T_i \quad (3.15)$$

In the β^- decay case

$$Q = T_f - T_i = T_e + T_\nu + T_D - T_P \approx T_e + T_\nu \quad (3.16)$$

³Positron annihilation can also produce 3 γ s instead of two. The latter process has lower probability with respect to 2γ production in proportion of 1 to 372. Which of the two decays happens depends on the relative spin orientation of the electrons.

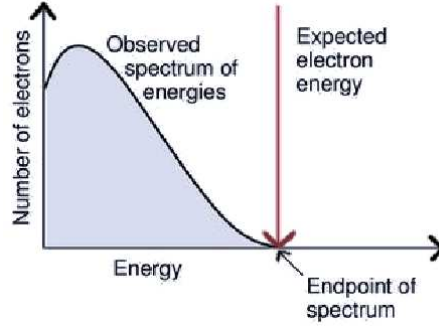


Figure 3.2: Electron spectrum from β decay. The endpoint of the spectrum is the highest energy the electron can have given momentum-energy conservation. The spectrum is a continuum of energies since the electron and the neutrino share all the possible combinations of momentum-energy allowed by the conservation principles.

T_D is the daughter kinetic energy while T_P is the parent one and since

$$m_P \approx m_D \gg m_e \gg m_\nu, \quad (3.17)$$

we can set them to zero with very good approximation. From the energy balance in Eq. 3.14 we have:

$$m(Z, A)c^2 - m(Z + 1, A)c^2 - m_e c^2 - m_\nu c^2 = T_e + T_\nu \quad (3.18)$$

and therefore:

$$T_e + T_\nu = m(Z, A)c^2 - m(Z + 1, A)c^2 - m_e c^2 \quad (3.19)$$

where m_e is the electron mass and we have neglected the extremely small neutrino mass m_ν . Since experimentally we deal with atoms and not with bare nuclei, we have to introduce the atomic mass:

$$m_{atom}(A, Z)c^2 = Zm_e c^2 + m(A, Z)c^2 - BE_{el}(Z) \quad (3.20)$$

where $BE_{el}(Z)$ is the binding energy of Z electrons. Inserting this definition in Eq. 3.19 we obtain:

$$\begin{aligned} Q = T_e + T_\nu = & \\ & m_{atom}(A, Z)c^2 - Zm_e c^2 + BE_{el}(Z) - m_{atom}(A, Z + 1)c^2 \\ & + (Z + 1)m_e c^2 - BE_{el}(Z + 1) - m_e c^2 = \\ & m_{atom}(A, Z)c^2 - m_{atom}(A, Z + 1)c^2 \end{aligned} \quad (3.21)$$

What we obtained for the β^- Q value is:

$$Q = [m_{atom}(A, Z) - m_{atom}(A, Z + 1)]c^2 \quad (3.22)$$

Since the decay can happen spontaneously only if $Q > 0$, this implies that the atomic mass of the parent must be larger than the one of the daughter.

3.5 Gamma Decay

We still have to introduce concepts about the nuclear structure, but a nucleus is a quantum many-body system like an atom. This means that nucleons occupy orbitals like electrons orbiting around the nucleus. In complete analogy with atoms, when a transition inside a nucleus happens, radiation can be emitted. In atoms, transitions result in visible light, IR, UV or X-ray emission. In nuclei the energies are higher (MeV range instead of the eV in atoms) and we can expect the emission of more energetic photons, the so called γ rays. The emission of a γ ray does not change the number of nucleons in the nucleus, so A and Z stay constant:



The symbol “*” indicates that the nucleus M is in an excited (higher) energy state: the de-excitation happens via emission of a γ ray. An example of γ decay is:



with a half-life of 250 days. The symbol “m” after the mass number indicates that Silver is in a **metastable state**. This means that the state is stable but it is not the one with minimum energy and there is a probability for the nucleus to decay. Metastable states are also called **isomers** of the stable nucleus. Metastable states have in general a longer lifetime (orders of magnitude) than excited states which have a “prompt” γ decay.

γ decays are classified in three categories:

- **Pure γ emission.** This is the case when a metastable isomer decays in a lower energy state emitting a γ ray. The γ energies are in the range between few keV and about 7MeV. The energy of the γ corresponds almost to the energy difference between the two levels since the recoil energy is small.
- **Internal Conversion.** The conversion happens when a γ ray is emitted and absorbed by an atomic electron. The electron is expelled from the atom after the absorption. A detector would measure an electron and no γ rays. The energy of the internal conversion electron will be the difference between the energy of the nuclear transition and the binding energy of the electron. In general, more electrons (called **Auger electrons**) or X-rays can be emitted after the expulsion of the first electron. This happens because after the conversion, the shell structure of the atom has a vacancy and the subsequent rearrangement for “filling the gap” generates more soft radiation or additional electrons. The name “Auger electrons” comes from **Pierre Auger (1899-1993)** which first observed this effect (see Fig. 3.3). Typically, Auger electrons are coming from the deepest orbitals (closer to the nucleus).
- **Pair Production.** This decay mode is quite uncommon: the nucleus de-excitation happens via the emission of an electron-positron pair. This can happen only if the transition energy is bigger than 1.02 MeV, which corresponds to twice the electron mass. The positron can undergo annihilation as discussed previously in the case of ${}^{22}\text{Na}$.

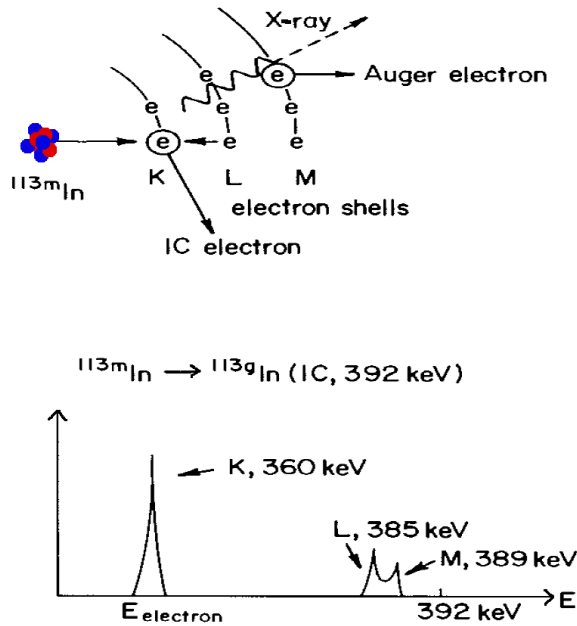
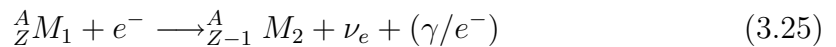


Figure 3.3: Internal conversion process: the example of ^{113}In . The energy spectrum below shows a sharp peak corresponding to the electron knocked out by the γ ray while the smaller peak correspond to Auger electrons following the rearrangement of the atomic shells.

3.6 Electron Capture

In this decay mode, discovered in 1938, an atomic electron is captured by the nucleus and an inverse β decay happens:



Electron capture decreases Z by one unit keeping A constant, like the β^+ decay. The expression (γ/e^+) symbolizes a variety of phenomena involving the emission of electrons (like Auger electrons) and/or radiation (X-rays and γ). An example of electron capture is the rare earth process:



Electron capture happens more often in high- Z nuclei and the electrons of the innermost shells are affected. Shells with principal quantum number $n=1$ are also called K-shells. When one of the electrons with $n=1$ is captured, the term K-capture is used. The next L-shells (with $n=2$) are also affected but this happens with a higher probability in heavy nuclei. Neutrinos emitted in this process have a precise energy equal to the difference between transition energy and electron binding energy. The binding energy is much smaller (eV range) and can be neglected with respect to the transition energy (MeV range).

3.7 Other Decays

The most common decays were discussed in the previous sections, while here we list some less common decays which can happen.

- **Spontaneous Fission.** Some nuclei, especially the very heavy ones, might undergo spontaneous fission. This means the the parent nucleus splits in two daughter nuclei of similar weight. An example is ${}_{98}^{252}\text{Cf} \longrightarrow {}_{38}^{98}\text{Cf} + {}_{60}^{152}\text{Ne} + 2n$. Spontaneous fission is different from induced fission which will be treated later.
- **Delayed Neutron Emission.** Some nuclei, after β decay can form daughter nuclei in excited states. Usually these states decay γ but in some cases, a neutron is emitted right after the β decay. The lifetime of the neutron emission is about the same of the previous β decay. This happens usually to neutron-rich fission products.
- **Delayed Proton Emission.** This decay is the analogous of the previous one and is even more rare. It is likely to happen in proton-rich nuclei.
- **Double- β Decay** Some nuclei are stable against β decay but not against double β decays. This decay is more rare than the normal β decay and it consists of the simultaneous (not sequential) emission of two electrons. An example is the decay ${}_{34}^{82}\text{Se} \longrightarrow {}_{36}^{82}\text{Kr} + 2e^{-} + 2\bar{\nu}_e$ with a half-life of about 10^{20} years. The long half-life underlines the rarity of this process⁴.
- **Double Proton Decay** This is the proton version of the previous decay. Predicted in the 70s, it was first observed in 1983 in the proton-rich nuclei ${}^{22}\text{Al}$ and ${}^{26}\text{P}$.
- **Cluster Emission Decay.** In this decay, an unstable nucleus decays expelling a cluster of protons and neutrons. The cluster is a tightly bound nucleus. The α decay is in some sense an example. The expulsion of an ${}^{14}\text{C}$ is the case of ${}^{222}\text{Ra}$ and ${}^{224}\text{Ra}$. Emission of ${}^{24}\text{Ne}$ from ${}^{232}\text{U}$ was also observed. What distinguishes cluster emission from fission is the fact that the cluster is significantly lighter than the parent nucleus.

3.8 Decay Combinations and Branching Ratios

In this chapter, we discussed the possible decay modes of a radioactive nucleus. The decay can significantly alter the nuclear structure, changing an element into another but sometimes there are also consequences at the atomic (and therefore chemical) level. Some nuclei can have more than one decay mode: all the allowed decays might happen with different probabilities which are called **branching ratios**. An example

⁴Another version is the so-called neutrinoless double β decay where there are two electrons in the final state but no neutrinos. This process has not yet been observed and it is subject to active research at the interface between nuclear and particle physics. If found, it would be a major discovery since it is linked to still unknown neutrino properties.

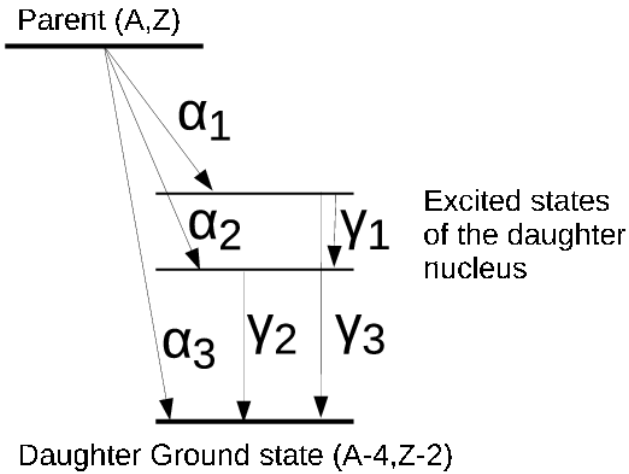


Figure 3.4: Possible α decays to different daughter states and possible following γ decays.

is the transuranic element ^{252}Cf which can either α -decay or undergo spontaneous fission. The branching ratios for the two decays are 97% and 3% respectively.

Another frequent case is the combination of different decays. For example, a nucleus can undergo α decay from a nucleus (A,Z) to a daughter (A-4,Z-2) or to an excited state of the same daughter. The excited state will decay to the ground state with γ emission, see Fig. 3.4;

3.9 Rates of Decay

The radioactive decay is a random process: the time when an unstable nucleus will decay is a random variable. If we observe a sample containing a radioactive substance and therefore many radioactive nuclei, we can draw some statistical results although the behavior of the single nucleus is not deterministically predictable.

The rate of decay (the number of decays per unit time) is called **activity \mathbf{A}** (measured in Becquerels (Bq) or disintegrations per second).

3.9.1 Single Decay

In differential notation, considering an infinitesimal time interval, the activity is $A = -dN(t)/dt$ where N is the number of nuclides at the time t . The minus sign comes from the fact that the amount of nuclei must decrease in time. The amount of disintegrations per unit time must be proportional to the amount of radioactive nuclei present at a certain time (more nuclei implies more decays) and therefore we can write:

$$-\frac{dN}{dt} = \lambda N \quad (3.27)$$

Given the boundary condition $N(0) = N_0$ it is easy to solve the last differential equation by variable separation. The solution is

$$N = N_0 e^{-\lambda t} \quad (3.28)$$

The inverse of λ has the dimension of time: $\tau = 1/\lambda$ and it is called **average lifetime** of the radioactive nucleus. For calculating the time after which the initial amount of radioactive material N_0 is reduced by one half ($N_0/2$), we ask:

$$\frac{N_0}{2} = N_0 e^{-\lambda t_{1/2}} \quad (3.29)$$

and solve for $t_{1/2}$:

$$t_{1/2} = \tau \ln 2 \approx 0.693\tau \quad (3.30)$$

The lifetime $t_{1/2}$ is called **half-life** of the radioactive nucleus.

3.9.2 Branching Decay

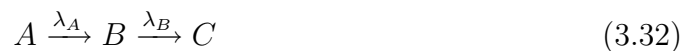
A parent nucleus can have multiple decay channels. Let's consider a nucleus that can decay via two different decays each of them having different probabilities and therefore different average decay times τ_1 and τ_2 . The total activity A of the nucleus will be composed by the sum of the activities of both decay channels (since they are independent):

$$A = \frac{1}{\tau_1} N + \frac{1}{\tau_2} N = N \left(\frac{1}{\tau_1} + \frac{1}{\tau_2} \right) = \frac{N}{\tau} \quad (3.31)$$

where N_1 and N_2 are the amounts of the two possible daughters resulting from the two possible decays. The average time τ is the total average time of the parent nucleus.

3.9.3 Double Decay Chain

A parent nucleus A can decay in another radioactive nucleus B which decays into C . Both decays will have an inverse average time λ_A and λ_B respectively:



The parent A decays according to the already known formula $-\frac{dN_A}{dt} = \lambda_A N_A$. B is also decaying (into C) and therefore $-\frac{dN_B}{dt} = \lambda_B N_B$ is also true. The amount of B nuclei depend from the decay of A and actually the decay rate of A is equal to the formation rate of B :

$$\frac{dN_B}{dt} = \lambda_A N_A \quad (3.33)$$

Note that the minus sign is not present in the last equation, since we are considering formation, not decay. Combining the equations we have so far, the overall change in the activity of B is:

$$\frac{dN_B}{dt} = \lambda_A N_A - \lambda_B N_B \quad (3.34)$$

If we suppose to know the initial amount of A (N_A^0) we can solve the first activity differential equation obtaining $N_A = N_A^0 e^{-\lambda_A t}$. Inserting this result in Eq. 3.34 gives the following differential equation:

$$\frac{dN_B}{dt} + \lambda_B N_B - \lambda_A N_A^0 e^{-\lambda_A t} = 0 \quad (3.35)$$

The solution of the last equation gives an expression for calculating the amount of B at any time, given the knowledge of the initial amount of A and of the two decay times:

$$N_B = \frac{\lambda_A}{\lambda_B - \lambda_A} N_A^0 (e^{-\lambda_A t} - e^{-\lambda_B t}) + N_B^0 e^{-\lambda_B t} \quad (3.36)$$

The last equation can be simplified in some limiting cases:

- **Transient Equilibrium:** Suppose the half-life of the parent A is longer than the one of the daughter B but still short enough that we can measure decays of A during an experiment. In this situation, after some time the term $e^{-\lambda_B t}$ can be ignored and Eq. 3.36 reduces to

$$N_B = \frac{\lambda_A}{\lambda_B - \lambda_A} N_A^0 e^{-\lambda_A t} = \frac{\lambda_A}{\lambda_B - \lambda_A} N_A \quad (3.37)$$

or, rearranging the terms:

$$\frac{N_B}{N_A} = \frac{\lambda_A}{\lambda_B - \lambda_A} \quad (3.38)$$

The last formula tells us that after some half-lives of the daughter nucleus B, the ratio of the two species reaches a phase where it is constant. Since the activity is proportional to the number of nuclei present, also the ratio of the activities becomes constant.

- **Secular Equilibrium:** In this case, if the half-life of the parent is much longer than that of the daughter (so long that we will not detect decays of A during an experiment), then $e^{-\lambda_B t} \approx 0$ and $\lambda_B - \lambda_A \approx \lambda_B$. Eq. 3.38 becomes:

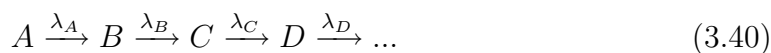
$$\frac{N_B}{N_A} = \frac{\lambda_A}{\lambda_B} \Rightarrow N_B \lambda_B = N_A \lambda_A \Rightarrow A_A = A_B \quad (3.39)$$

After a sufficiently long time, the activity of the parent and of the daughter become equal.

If the lifetimes of A and B are comparable, no approximation is valid and no equilibrium situation exists.

3.9.4 Multiple Decay Chain

We can consider an arbitrarily long decay chain:



The first two steps were treated in the previous section. The third step will depend on the previous one in the same way:

$$\frac{dN_C}{dt} = \lambda_B N_B - \lambda_C N_C \quad (3.41)$$

The final equation for the activity of C will be a more complicated version of Eq. 3.36:

$$N_C = -\lambda_C N_C + \lambda_B \left[\frac{\lambda_A}{\lambda_B - \lambda_A} N_A^0 (e^{-\lambda_A t} - e^{-\lambda_B t}) + N_B^0 e^{-\lambda_B t} \right] \quad (3.42)$$

and so on.

A way to simplify the problem in the case of N nuclei in the chain is due to H. Bateman (the **Bateman equation**). Suppose we have a chain of N radioactive nuclei with inverse average decay times λ_i with $i=1..N$. Let's assume also that only the parent is present at the beginning: $N_1(0) = N_1^0$ and $N_i(0) = 0$ for $i \geq 2$. Under the above hypotheses, the abundance of the final nucleus N_N is

$$N_N = C_1 e^{-\lambda_1 t} + C_2 e^{-\lambda_2 t} + \dots = \sum_{i=1}^N C_i e^{-\lambda_i t} \quad (3.43)$$

and the constants C_i are:

$$C_1 = \frac{\prod_{i=1}^{N-1} \lambda_i}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1) \dots (\lambda_N - \lambda_1)} N_A^0 \quad (3.44)$$

$$C_2 = \frac{\prod_{i=1}^{N-1} \lambda_i}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2) \dots (\lambda_N - \lambda_2)} N_A^0 \quad (3.45)$$

and so on for $C_3..C_N$. The numerator of the coefficients C is the product of all the decay constants except the last one. The denominator is the product of all the decay constants with the one of interest subtracted. Each coefficient contains also the initial amount of the parent N_A^0 .

3.10 Creation of Nuclides by Neutron Flux Irradiation

Neutrons can activate a nucleus A transforming it into a radioactive one B. Nuclear reactors are an example of environment where a high neutron flux ϕ (number of neutrons per area per time) is present. If the cross section for neutron capture is σ , and the number of target nuclei is n, then the formation rate R of the new nucleus B is

$$R = n\phi\sigma \quad (3.46)$$

Usually, n, ϕ and σ are constants, so R is also constant. If the decay constant of B is λ_B , its activity is:

$$\frac{dN_B}{dt} = n\phi\sigma - \lambda_B N_B \quad (3.47)$$

The solution of the above equation, giving the total activity after neutron irradiation is:

$$A_B = \lambda_b N_B = n\phi\sigma(1 - e^{-\lambda_B t}) \quad (3.48)$$

Chapter 4

Nuclear Force and Nuclear Structure

4.1 Introduction

The nucleus is a many-body quantum system and as such it is extremely difficult to describe. There are some analogies with the difficulties in modeling atoms but also differences complicating more the physics of nuclei. The electrons around an atom experience with very good approximation a predominant central Coulomb force from the nucleus plus a weaker force coming from the repulsion among the electrons themselves. The force is purely electromagnetic and completely known.

In a nucleus, there is no clear center from where the confining force comes from: all the nucleons are exerting a force to all the others. The force keeping nucleons bound is called nuclear force and it will be discussed more in details later. This force is still not fully known. For complicating things more, there are two different particles to take into account: protons and neutrons. All the latter features give the nucleus a very rich phenomenology.

Roughly, nuclear models can be divided in two main categories: **collective models** and **microscopic models**. Collective models disregard the particle structure and try to treat the nucleus as a whole. This means that collective coordinates (mass, volume, etc.) are used. Microscopic models are more fundamental and try to describe the nucleus starting from its proton-neutron structure with different degrees of approximation. In this case, the coordinates are positions, momenta, spin etc. of the single particles. We will discuss collective models first since a detailed knowledge of the nuclear force is not needed.

4.2 Weizsäcker Semiempirical Mass Formula

The model takes its name from the name of its proposer **Carl von Weizsäcker (1912-2007)**. Experimentally, the binding energy $E_B(Z,N)$ of a nucleus is approximately proportional to the number of nucleons A . This means that the binding energy is also proportional to the volume of the nucleus: this is why such a model is

also referred as liquid drop model. The nucleus is thought as an incompressible liquid drop with a certain volume (proportional to the particles contained) and surface. In first approximation we can write

$$E_B(Z, N) = \alpha_1 A \quad (4.1)$$

The coefficient α_1 is called volume energy parameter. The last equation is very approximate and needs corrections since it tends to overestimate the binding energy as A grows. The first correction comes from surface effects: nucleons at the surface of the nucleus are less bound so the total binding energy should decrease with the number of them:

$$E_B(Z, N) = \alpha_1 A - \alpha_2 A^{2/3} \quad (4.2)$$

and α_2 is the surface energy parameter.

Another corrections comes from the electric repulsion among protons which should diminish the total binding energy:

$$E_B(Z, N) = \alpha_1 A - \alpha_2 A^{2/3} - \alpha_3 \frac{Z^2}{A^{1/3}} \quad (4.3)$$

with α_3 the Coulomb energy parameter. In the third term, the formula of the Coulomb energy can be recognized. The $1/A^{1/3}$ factor comes from the fact that we are considering the nucleus a spherical object with constant density and therefore the radius is $R = r_0 A^{1/3}$. All the constants are factorized in the Coulomb energy parameter. According to this term, isobaric (same A) nuclei with less protons are more bound.

Another correction comes from the following observed fact: nuclei with $Z \approx N$ are more bound. An imbalance among neutrons and protons corresponds to a less bound nucleus. This observation leads to the symmetry energy correction:

$$E_B(Z, N) = \alpha_1 A - \alpha_2 A^{2/3} - \alpha_3 \frac{Z^2}{A^{1/3}} - \alpha_4 \frac{(Z - N)^2}{A} \quad (4.4)$$

with α_4 the symmetry energy parameter. The presence of a factor A in the denominator compensates the growing number of neutrons present in heavy nuclei. In heavy nuclei, the electric charge is so high that only more neutrons can compensate for it and keep the nucleus bound, therefore a larger asymmetry among nucleons is tolerated.

The existence of the symmetry term already points to certain properties of the nuclear force. Another property is that even- Z /even- N nuclei (even-even for short) are more bound. This property is called *pairing*. For dealing with pairing force, we introduce another correction:

$$E_B(Z, N) = \alpha_1 A - \alpha_2 A^{2/3} - \alpha_3 \frac{Z^2}{A^{1/3}} - \alpha_4 \frac{(Z - N)^2}{A} + \Delta \quad (4.5)$$

where

$$\Delta = \begin{cases} \delta & \text{for even-even nuclei} \\ 0 & \text{for odd-mass nuclei} \\ -\delta & \text{for odd-odd nuclei} \end{cases} \quad (4.6)$$

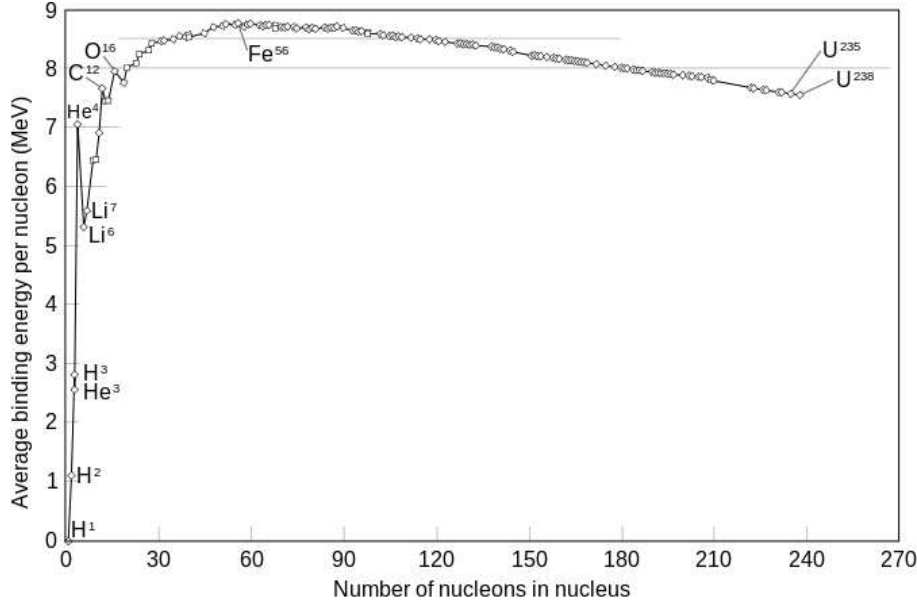


Figure 4.1: Binding energy per nucleon as a function of the number of nucleons A .

The last final form obtained is the Weizsäcker mass formula. Typical values for the constants are (depending on the fitted dataset):

$$\begin{aligned}
 \alpha_1 &= 16 \text{ MeV} & \alpha_2 &= 17 \text{ MeV} \\
 \alpha_3 &= 0.6 \text{ MeV} & \alpha_4 &= 25 \text{ MeV} \\
 \delta &= \frac{25}{A} \text{ MeV} & & .
 \end{aligned}
 \tag{4.7}$$

The parameter δ is the less determined one since it varies significantly more than the others given the dataset considered.

In Fig. 4.1 is shown the binding energy per nucleon as a function of the number of nucleons and a fit using the Weizsäcker mass formula. Note that E_B/A grows towards a maximum around the Fe nucleus ($A \approx 56$) and then slowly decreases (because of Coulomb repulsion). This behavior explains why we can obtain energy by nuclear fusion with light elements or nuclear fission with heavy elements. If two light elements fuse together, the E_B/A will be higher for the resulting nucleus and the excess energy will be released. The opposite happens with heavy nuclei: for them it is more energetically convenient to fission and produce two daughter nuclei with higher E_B/A .

4.3 Vibrational Model

Following the analogy with a liquid drop, in first approximation we can consider a nucleus as a sphere. In this way, for the moment, we can neglect rotations. If the nucleus is excited (for example with radiation of by another particle), it might start vibrating around the ground-state spherical shape. If the vibration is not too big, the density of the nucleus will stay the same but the shape of the surface will vary.

We can describe the departure from the spherical shape with

$$R(\theta, \phi, t) = R_0 \left\{ 1 + \sum_{\lambda\mu} \alpha_{\lambda\mu}(t) Y_{\lambda\mu}(\theta, \phi) \right\} \quad (4.8)$$

where R is the distance from the center to the surface at a given point (in spherical coordinates) at a given time t . R_0 is the equilibrium radius (radius of the ground-state sphere). The parameters $\alpha_{\lambda\mu}$ are called **shape parameters**.

Each oscillation mode λ has $2\lambda + 1$ values for μ . Symmetry requirements and the fact that R should be a real number reduce the number of combinations allowed.

The $\lambda = 1$ mode corresponds to an oscillation around a fixed point or to an oscillation of the protons against the neutrons (the “giant resonance”, see next). The $\lambda = 2$ mode is called **quadrupole mode** and the $\lambda = 3$ **octupole mode** and so on.

The rate of change of the nucleus can be thought of as a velocity, therefore we can write the “kinetic energy” as

$$T = \frac{1}{2} \sum_{\lambda\mu} A_\lambda \left| \frac{d\alpha_{\lambda\mu}}{dt} \right|^2 \quad (4.9)$$

where B_λ might be interpreted as a mass of the liquid drop. Continuing the non-relativistic analogy to a vibrational problem, we might introduce also a potential energy:

$$V = \frac{1}{2} \sum_{\lambda\mu} B_\lambda |\alpha_{\lambda\mu}|^2 \quad (4.10)$$

The parameters B_λ can be related to the surface and Coulomb energies of the liquid drop (see also the previous model). The equations of motion are therefore:

$$A_\lambda \frac{d^2 \alpha_{\lambda\mu}}{dt^2} + B_\lambda \alpha_{\lambda\mu} = 0 \quad (4.11)$$

The frequency of the oscillation is:

$$\omega_\lambda = \sqrt{\frac{B_\lambda}{A_\lambda}} \quad (4.12)$$

The minimum quantum of oscillation energy will be $E = \hbar\omega_\lambda$. A nucleus can transition from one vibrational state to another emitting radiation. The radiation energy will be equal to the energy difference among the levels.

4.4 Rotational Model

Previously, we assumed a basic spherical shape for a nucleus. There is no fundamental reason why a nucleus cannot have a different shape in its ground state and in fact this is what is realized in Nature: nuclei can have different shapes which differ from a sphere. Such nuclei are called **deformed**. In considering different shapes, we are forced also to consider the possibility of rotational degrees of freedom. We

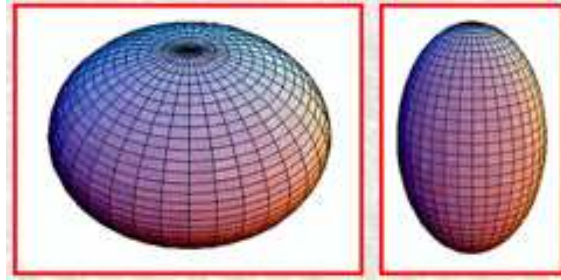


Figure 4.2: Graphic representation of an oblate (left) and prolate (right) nuclei.

are not going to discuss rotational models in detail but try to highlight the main observations.

In general, closed-shell¹ nuclei are spherical. The reason is that for closed-shell nuclei, the M quantum number (the projection of spin along the quantization axis) is zero. When $M=0$, the object is invariant under rotations and therefore the shape must be spherical. For nuclei with more or less nucleons than the closest closed-shell nucleus, it is energetically more convenient to assume a deformed shape. The deformations are mainly classified in two kinds, taking an axis (usually z) as reference (see Fig 4.2):

- **Prolate nuclei:** elongated along the z axis
- **Oblate nuclei:** flattened at the poles.

Stable nuclei are generally close to the spherical shape, while heavy nuclei ($A \geq 150$) show (large) deformations. Quantum mechanically rotational states of deformed nuclei can be described starting from the hamiltonian of a quantum rotator:

$$H = \sum_i^3 \frac{\hbar^2 J_i^2}{2I_i} \quad (4.13)$$

where the sum is over the three inertia axes, I_i are the moments of inertia and J_i the angular momentum operators. Starting from this point, eigenfunctions and eigenvalues can be calculated and confronted with experimental results.

4.5 Giant Resonance

The phenomenon of giant resonance is found in almost every nucleus which is excited with another particle (electron, photon, proton, another nucleus,..). TO fix the ideas, let's imagine a beam of protons hitting a ^{208}Pb nucleus and exciting it. using the reaction notation: $^{208}\text{Pb}(p,p')^{208}\text{Pb}^*$. If we measure the energy difference between the incoming and the emerging protons (p and p' respectively) and report the counts

¹In analogy with atoms, a closed-shell nucleus is a nucleus where all the orbitals are filled by nucleons without any "hole". For example, Helium, the lightest noble gas has the 1s and 2s orbitals filled. Helium is also "noble" in the sense of nuclear physics, since the first nuclear energy level is completely filled with two protons and two neutrons. The nuclear shell structure will be treated more in details later in this chapter.

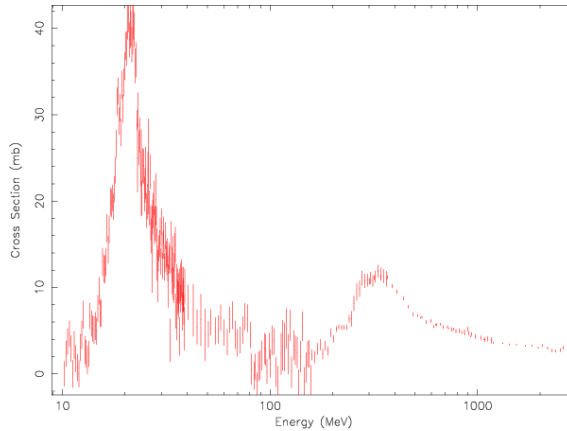


Figure 4.3: The first peak in the histogram is a giant dipole resonance.

as a function of this energy difference, we are looking at an excitation energy plot. The most striking feature of the plot is the appearance of a big resonance in the 10MeV region. Other resonances are all smaller and this happens for almost all nuclei. This prominent resonance is called **giant resonance** and it is a collective excitation of the whole nucleus. The energy E at which this resonance seats is well approximated by the empirical formula

$$E \approx 78 \times A^{-1/3} \quad (4.14)$$

The most common giant resonance (studied since 1940s) is called **giant dipole resonance** and its collective model interpretation is a movement of the protons against the neutrons (see Fig. 4.3). Since all the positive charges are moving as a whole, the nucleus is in first approximation like an electric dipole (and therefore the name).

4.6 Fermi Gas Model

The liquid drop model was assuming a nucleus composed by particles interacting strongly and it was somewhat a classical picture, although able to reproduce the general behavior of the binding energies. Now we would like to do a small step towards a quantum description of the nucleus. Actually, many properties of the nucleus can be derived thinking not at strongly interacting particles, but at non-interacting particles confined in a potential. The simplest of such models is the Fermi Gas Model. In this model, the nucleons move freely inside a sphere but they are subject to the Pauli exclusion principle (here is where quantum mechanics comes in). The sphere has radius $R = R_0 A^{1/3}$ with $R_0 \approx 1.2\text{fm}$. The confined nucleons, according to quantum mechanics can have only discrete energy levels. Protons and neutrons can be thought as confined in two separate potential wells. The wells are different in shape because the proton's one has the Coulomb barrier. In particular, the proton's well is higher than the one of the neutrons because of this. Each level of each well can be occupied at most by two nucleons (which differ from spin) because

of the Pauli exclusion principle. It is assumed that the temperature of the system is so low that the nucleons try to occupy the lowest levels possible up to a maximum kinetic energy E_F , called Fermi energy. The uncertainty principle $\Delta x \Delta p \leq \hbar$ tells us that the phase space must be thought as partitioned in cells of $h = 2\pi\hbar$ volume and therefore, the number of particles fitting in a phase space with volume $\int d^3x d^3p$ is

$$N = \frac{\int d^3x d^3p}{(2\pi\hbar)^3} = \frac{V \int d^3p}{(2\pi\hbar)^3} \quad (4.15)$$

where in the second equation we considered the total volume of physical space equal to V . For the second integral, we can consider spherical coordinates in the momentum space where the radius is the modulus of the momentum. Integrating over all the angles and up to a maximum radius p :

$$N = \frac{V p^3}{6\pi^2 \hbar^3} \quad (4.16)$$

Having assumed a spherical nucleus, $V = 4\pi R^3/3 = 4\pi R_0^3 A/3$. Combining the above formulas, for a neutron the maximum momentum will be

$$p_F(N) = \frac{\hbar}{R_0} \left(\frac{9\pi N}{4A} \right)^{1/3} \quad (4.17)$$

and for a proton:

$$p_F(Z) = \frac{\hbar}{R_0} \left(\frac{9\pi Z}{4A} \right)^{1/3} \quad (4.18)$$

From the momentum, it is possible to calculate the Fermi energy:

$$E_F = \frac{p^2}{2m} \approx 40 \text{ MeV} \quad (4.19)$$

The mean energy of a nucleon can be obtained with:

$$\langle E \rangle = \frac{\int_0^{p_F} E d^3p}{\int_0^{p_F} d^3p} = \frac{3}{5} E_F \approx 24 \text{ MeV} \quad (4.20)$$

The last result shows that the average energy per nucleon is quite smaller with respect to its rest mass and this justifies a non-relativistic treatment of the nucleus, at least with good approximation.

Using the last expressions, the total average kinetic energy considering both protons and neutrons is

$$\langle E(Z, N) \rangle = Z \langle E_Z \rangle + N \langle E_N \rangle = \frac{3}{10m} (Z p_F^2(Z) + N p_F^2(N)) \quad (4.21)$$

and therefore:

$$\langle E(Z, N) \rangle = \frac{3}{10m} \frac{\hbar^2}{R_0^2} \left(\frac{9\pi}{4} \right)^{2/3} \frac{N^{5/3} + Z^{5/3}}{A^{2/3}} \quad (4.22)$$

The latter energy expression has a minimum for an equal number of protons and neutrons $N=Z=A/2$ for a given A .

Let's try to understand how the energy varies in the neighborhood of the minimum assuming $Z+N=A$ and $Z-N=\epsilon$ where ϵ is a small number. Inverting the latter relations, $Z = \frac{1}{2}A(1 + \epsilon/A)$ and $N = \frac{1}{2}A(1 - \epsilon/A)$. Now $x = \epsilon/A$ is a small parameter and we can use the Taylor expansion $(1+x)^n \approx 1 + nx + \frac{n(n-1)}{2}x^2 + \dots$ in Eq. 4.22 obtaining

$$\langle E(Z, N) \rangle = \frac{3}{10m} \frac{\hbar^2}{R_0^2} \left(\frac{9\pi}{4} \right)^{2/3} \left(A + \frac{5}{9} \frac{(Z-N)^2}{A} + \dots \right) \quad (4.23)$$

We notice the close similarity with the liquid drop model: the leading term is proportional to A (as the volume energy) while the next term has the same form as the symmetry energy.

It is also possible to derive a surface energy correction by subtracting to the full energy a term obtained integrating only the momenta on Fermi surface. Such a term is negligible in the large volume limit.

4.7 Magic Numbers and the Shell Model

We have seen already in the over-simplified Fermi gas model, that nucleons can be treated as independent particles inside a potential well. The Pauli exclusion principle permits to avoid "collisions" among nucleons inside the nucleus and this is why an independent particle model approximately works. In the following, we are going to discuss the so-called **Shell Model** of the nucleus, which is the first step towards a microscopic description of nuclei.

Already in the 1930s it was recognized that nuclei with Z and N (or both) having values like **2, 8, 20, 28, 50, 82, 126** were particularly stable. Those numbers were called **magic numbers**.

Magic nuclei are ${}^4\text{He}$, ${}^{16}\text{O}$, ${}^{40}\text{Ca}$, ${}^{90}\text{Zr}$, ${}^{208}\text{Pb}$ with $Z=2,8,20,40,82$ and $N=2,8,20,50,82,126$.

From the beginning, J.Bartlett and W.Elsasser in 1932 tried to explain the pattern of the magic numbers with a shell model analogous to the atomic one, but they were not able to explain all of them. It was only in the early 1950s that M.Goeppert-Mayer and J.Jensen were able to explain all the magic numbers: the key missing element was a **spin-orbit coupling** in the nuclear force.

Basic facts pointing to the existence of magic numbers are:

- The energies of the first excited states in magic nuclei are higher than those of nearby nuclei (see Fig. 4.4).
- The energy needed for removing a nucleon from a magic nucleus is higher than in nearby even-even nuclei.
- The shape of the ground state is spherical for magic nuclei.

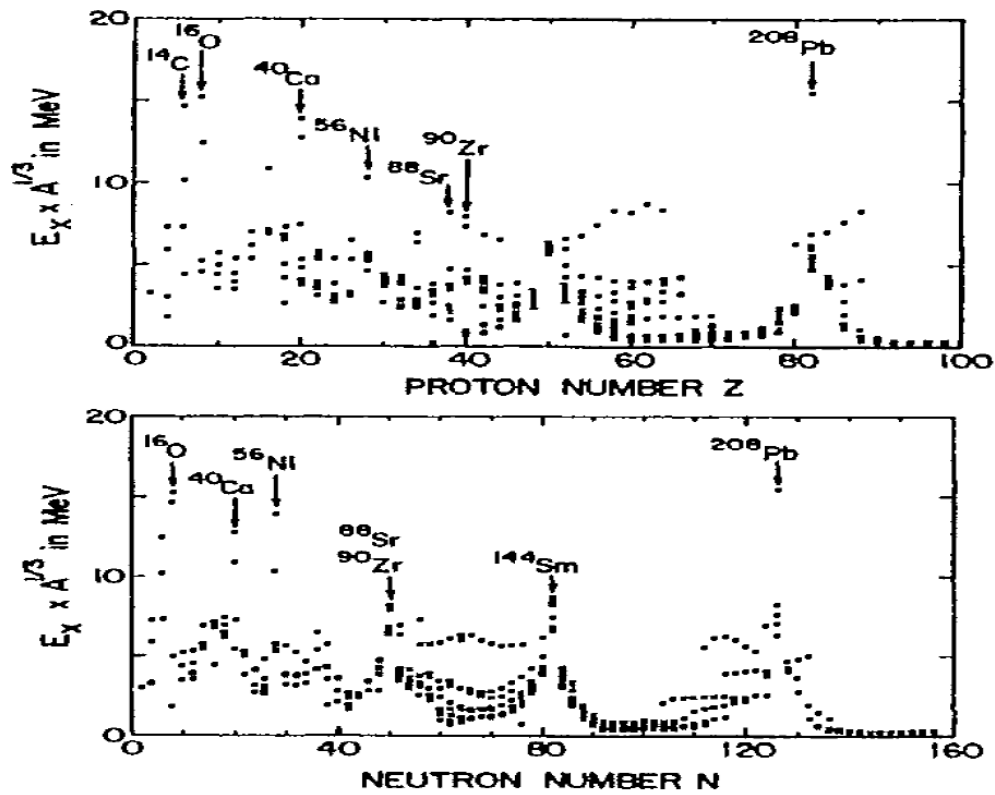


Figure 4.4: Energy of the first excited states as a function of the proton (top) and neutron (bottom) numbers. The diagram clearly shows peaks at the location of the magic numbers.

4.7.1 Digression: The Nuclear Quantum Many-body Problem

In the following, we will only introduce the formalism for microscopic models since a full treatment would bring us too far from the topic of these notes. Moreover, it is a very complex topic, still subject of active research.

The one of the central problems is the solution of the N-body stationary Schroedinger equation:

$$H\Psi_\alpha(r_1, r_2, \dots, r_A) = E_\alpha\Psi_\alpha(r_1, r_2, \dots, r_A) \quad (4.24)$$

where H is the hamiltonian, E_α the energy eigenvalues and $\Psi_\alpha(r_1, r_2, \dots, r_A)$ the N-body eigenfunctions dependent from the coordinates of every nucleon. The hamiltonian is composed by a kinetic and a potential term:

$$H = \sum_{i=1}^A \frac{\hbar^2}{2\mu_i} \nabla_i^2 + \sum_{i \neq j} V_{ij} \quad (4.25)$$

where μ_i is the reduced mass of the nucleons. The usual approach is the expansion of the eigenfunctions on a complete basis state up to D states:

$$\Psi_\alpha(r_1, r_2, \dots, r_A) = \sum_{k=1}^D C_k^\alpha \Phi_k(r_1, r_2, \dots, r_A) \quad (4.26)$$

Multiplying Eq. 4.24 from the left by Φ^* , using the definition 4.26 and the completeness of the basis the problem can be recast in the following matrix form:

$$\sum_{k=1}^D H_{jk} C_k^\alpha = E_\alpha C_j^\alpha \quad (4.27)$$

The energy eigenvalues can be obtained solving the secular problem $\det(H - E_\alpha I) = 0$. Once the E_α are obtained, also the expansion coefficients C_j^α are known and therefore also the wavefunctions. It is clear that for a correct description of the nucleus, a large value of D should be used, which complicates the numerical treatment of the problem. A careful choice of the expansion basis is also crucial for the feasibility of the calculation.

We are not going into the details of the basis states, but we just point out that since the nucleons are Fermions (particles obeying the Fermi statistics and the Pauli exclusion principle) the basis states must be antisymmetric. The correct antisymmetrization is achieved writing the basis state as a *Slater determinant* of single-particle wavefunctions:

$$\Phi_k(r_1, r_2, \dots, r_A) = \frac{1}{\sqrt{A!}} \det \begin{vmatrix} \phi_1(r_1) & \phi_1(r_2) & \dots & \phi_1(r_A) \\ \phi_2(r_1) & \phi_2(r_2) & \dots & \phi_2(r_A) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_A(r_1) & \phi_A(r_2) & \dots & \phi_A(r_A) \end{vmatrix} \quad (4.28)$$

A simple choice for the single-particle wavefunctions are the harmonic oscillator ones.

4.8 The Nuclear Force

Chapter 5

Nuclear Reactions

Nuclear reactions are a general name for the interactions of nuclei with other particles or other nuclei. Nuclear reactions are important both in applications and in basic science as a method to explore the nuclear structure.

5.1 Reaction Types

Reactions are classified mainly in two groups:

- **Elastic Reactions:** in this case the target nucleus and the projectile do not change at the end of the reaction. For example, proton elastic scattering against carbon will be: $^{12}\text{C}(p,p)^{12}\text{C}$.
- **Inelastic Reactions:** in this case, the projectile loses energy which goes into the target for exciting or breaking it. In the latter case of proton scattering against carbon we could have: $^{12}\text{C}(p,p')^{12}\text{C}^*$.

If we consider the inelastic collision of two nuclei, more sub-cases arise. Let's consider the collision of Deuterium as projectile and Cobalt as target. Some possible cases are summarized below:

Reaction	Type
$^{59}\text{Co}(d,d)^{59}\text{Co}$	Elastic
$^{59}\text{Co}(d,d')^{59}\text{Co}^*$	Inelastic
$^{59}\text{Co}(d,\gamma)^{61}\text{Ni}$	Radiative Capture
$^{59}\text{Co}(d,p)^{60}\text{Co}$	Stripping
$^{59}\text{Co}(d,n)^{60}\text{Ni}$	Stripping
$^{59}\text{Co}(d,^3\text{He})^{58}\text{Fe}$	Pickup
$^{59}\text{Co}(d,\alpha)^{57}\text{Fe}$	Pickup

During a nuclear reaction, energy, linear momentum, angular momentum and mass number are conserved. The amount of energy absorbed or released by the reaction is commonly known as the **Q** of the reaction. If **Q** is positive, the reaction is **exoergic** and energy is released. If the **Q** is negative, the reaction is **endoergic** and the

reaction requires energy in order to happen.

If the Q is negative, the projectile must have a higher energy than $|Q|$ for initiating the reaction. The reason is that part of the projectile's energy goes into momentum conservation and in overcoming the Coulomb repulsive barrier.

the minimum amount of energy needed to initiate the reaction is called **threshold energy**.

5.2 Momentum Correction

We would like to estimate the amount of energy needed for initiate a reaction with negative Q . Let's consider an approximate case where we neglect relativity and assume the target nucleus at rest with mass M . The projectile with mass m will hit the target with velocity v_i in the center forming a compound nucleus with total mass $(m+M)$. After the collision, the final nucleus will move with velocity v_f .

The kinetic energies before and after the collision will be $E_i = \frac{1}{2}mv_i^2$ and $E_f = \frac{1}{2}(M+m)v_f^2$ respectively.

Momentum conservation requires:

$$mv_i = (m+M)v_f \Rightarrow v_f = \frac{m+M}{m}v_i \quad (5.1)$$

Squaring v_f in the last equation and comparing it with $v_i = 2E_i/m$ from the initial kinetic energy:

$$\frac{2E_i}{m} = \left(\frac{m+M}{m}\right)^2 v_f^2 \quad (5.2)$$

and solving for E_i we obtain:

$$E_i = \frac{(m+M)^2 v_f^2}{2m} \quad (5.3)$$

Now we can calculate the ratio between final and initial kinetic energies:

$$\frac{E_f}{E_i} = \frac{m}{m+M} \quad (5.4)$$

The ratio in the last equation represents the fraction of initial kinetic energy required to fulfill the momentum conservation law. We can now calculate $\frac{E_f}{E_i} + X = 1$: X will be the fraction of energy left and available for the reaction:

$$X = \left(\frac{M}{m+M}\right) E_i \quad (5.5)$$

The last fraction should be equal at least to Q for starting the reaction (the threshold energy), therefore we need a projectile energy of at least:

$$E_i^{th} = \left(\frac{m+M}{M}\right) |Q| \quad (5.6)$$

As an example application, consider the reaction (discovered by Rutherford, see the history chapter):



with $Q = -1.19$.

Using Eq. 5.6 we can calculate the minimum kinetic energy of the α particle needed:

$$E^{th} = \left(\frac{M_{\alpha+M_N}}{M_N} \right) Q = \left(\frac{14+4}{14} \right) 1.19 \approx 1.5\text{MeV} \quad (5.8)$$

The last estimate is still an underestimation: we have to take into account additional energy needed to overcome the Coulomb repulsion between the nuclei.

5.3 Coulomb Barrier Correction

The Coulomb barrier the projectile particle with Z_1 protons has to overcome for a target with Z_2 protons and a distance equal to the sum of their radii is:

$$E_c = \frac{Z_1 Z_2 e^2}{R_1 + R_2} \quad (5.9)$$

Remembering that $e^2 = 1.44\text{MeVfm}$ and using for the radius the uniform density approximation $R = R_0 A^{1/3} = 1.4A^{1/3}\text{fm}$:

$$E_c = \frac{1.44 \cdot Z_1 Z_2}{1.4(A_1^{1/3} + A_2^{1/3})} \quad (5.10)$$

In the case of the reaction considered before:

$$E_c = \frac{1.44 \times 2 \times 7}{1.4(14^{1/3} + 4^{1/3})} \approx 3.6\text{MeV} \quad (5.11)$$

The last estimation is classical and in some cases, quantum mechanical tunneling can lower the Coulomb barrier energy needed.

Combining the two corrections we calculated so far, we need a projectile with an energy of at least the Coulomb barrier energy times the fraction required by the momentum correction. Applying this to the αN reaction:

$$E^{th} = \left(\frac{M_{\alpha+M_N}}{M_N} \right) \times E_{Coulomb} = \frac{14+4}{14} \times 3.6 \approx 4.6\text{MeV} \quad (5.12)$$

So we need at least 4.6MeV α particles for initiating the reaction.

5.4 Cross Section

A key parameter describing a reaction is its **cross section** usually indicated with the letter σ . The cross section is what an experiment can measure and it is also what the theoretical calculations predict. The cross section is directly related to

the probability of the reaction to happen. Classically, the cross section has a simple geometrical meaning. Since the projectile, for interacting with the target must hit it, the probability of the process should be proportional to the area of the target seen by the projectile. In a very simple approximation where the nucleus is a sphere, the total geometric cross section will be $\sigma = \pi R^2$ where R is the radius of the nucleus. In fact, since the nucleus is a sphere, the effective surface the projectile “sees” is a disk of area σ . In nuclear physics, cross sections are of the order of one **barn** $=10^{-24}\text{cm}^2=100\text{fm}^2$. During an experiment, we can measure the number of particles emerging from a reaction in a given energy range and in a given angular range. This permits the calculation of a **differential cross section**. If we would be able to measure all the energies and all the angles, then we can calculate the **total cross section** which is the equivalent of the geometric cross section previously considered.

Let’s see how the cross section is experimentally measured in some cases.

5.4.1 Thin Target

A target is considered *thin* if the projectile particles are not attenuated passing through it. The number of particles per second I is called **flux (or current)** and the flux per area is called **flux density or current density** and is measured in $\text{cm}^{-2}\text{s}^{-1}$. If the target is thin, the flux density is constant across the target. The number of reactions induced by the projectiles per unit time (the rate R) is:

$$R = Id\rho\sigma \quad (5.13)$$

where d is the target thickness and ρ the number of target nuclei per unit volume. If the target is embedded into an uniform flux (the main example is a target inside the neutron flux of a nuclear reactor) the rate is given by

$$R = n\phi\sigma \quad (5.14)$$

Another useful form is obtained integrating over time the last one:

$$N_{\text{reactions}} = N_{\text{targets}}\phi\sigma T \quad (5.15)$$

where t is the total exposure time (irradiation time).

5.4.2 Thick Target

A target is *thick* if the projectiles are attenuated passing through it. If the initial flux is I_0 , the attenuation (difference in flux between two points in the target) will be proportional to the target thickness, the number of nuclei in the target, the intensity of the beam and the cross section. In differential terms, for a distance dx in the target, the attenuation will be:

$$-dI = I\rho\sigma \cdot dx \quad (5.16)$$

Solving for I with the boundary condition of an initial flux I_0 and starting distance $x=0$ we find:

$$I(x) = I_0e^{-\rho\sigma x} \quad (5.17)$$

Appendix A

Statistics and Error Analysis

A.1 Momentum Generating Functions

The momentum generating function $M_x(t)$ is defined as the expectation value of e^{tx} :

$$M_x(t) = E(e^{xt}) \quad (\text{A.1})$$

From the linearity properties of E we have:

$$M_x(t) = 1 + tE(x) + \frac{t^2}{2!}E(x^2) + \dots = \sum_i \frac{t^i}{i!}E(x^i) \quad (\text{A.2})$$

The momentum generating function gives all the moments M_k of the distribution according to:

$$M_k = \left. \frac{d^k M_x(t)}{dt^k} \right|_{t=0} \quad (\text{A.3})$$

A.2 Binomial Distribution

Let us consider a random event E which happens with a constant probability p. The probability that E will not happen is q=1-p. We would like to determine the probability P(N,k) that in N repeated trials the event E happens k times. In N trials, there are

$$\binom{N}{k} = \frac{N!}{(N-k)!k!} \quad (\text{A.4})$$

possible simple combinations (or groups of k positive outcomes). Since the events are independent, there is a probability $p^k q^{N-k}$ that we have k positive and (N-k) negative outcomes.

Putting together the latter observations, we have:

$$P(N, k) = \binom{N}{k} p^k q^{N-k} \quad (\text{A.5})$$

The average of this distribution is $E(x) = Np$ and the variance $V(x) = Npq$.

A.3 Poisson Distribution

The Poisson distribution can be seen as a limit of the Binomial distribution when $N \rightarrow \infty$ as the product Np stays constant, say $Np = \lambda$. Another way to look at the Poisson distribution is to say that it gives the probability to observe k events when λ are expected.

The limiting Binomial distribution can be rewritten as:

$$\lim_{N \rightarrow \infty} \binom{N}{k} p^k q^{N-k} = \frac{1}{k!} N(N-1)\dots(N-k+1) \frac{(Np)^k}{N^k} \left(1 - \frac{Np}{N}\right)^{N-k} \quad (\text{A.6})$$

The term $N(N-1)\dots(N-k+1)$ in the limit goes like N^k and the term in parenthesis as $e^{-\lambda}$. After substitution and simplifications we obtain the Poisson distribution:

$$P(\lambda, k) = \frac{\lambda^k e^{-\lambda}}{k!} \quad (\text{A.7})$$

The Poisson distribution can be obtained also asking what is the probability of a given number of events k occurring in a fixed interval of time (and/or space) if these events occur with a known average rate and independently of the time since the last event.

The parameter λ can be interpreted as the product of a rate R and a time interval T : $\lambda = RT$.

It turns out that $E(x) = Var(x) = \lambda$ so the Poisson distribution has average equal to its variance.

A.4 Exaple: Radioactive Decays

Let's consider a radioactive substance containing a radionuclide. The probability of detecting k decays in a certain time T is given precisely by the Binomial distribution changing the time T in a certain number of trial measurements N . In this case, the Poisson distribution is a very good approximation because we are measuring events with a small probability p but doing a very large number of observations N .

If we measure a rate R for the radioactive substance for a time T , we can calculate λ and answer the question about the probability of observing k events during the time T . In practice, whenever the Poisson distribution is a good approximation of the Binomial, the former is always preferred for computational reasons: the factorials in the Binomial distributions become quickly difficult to calculate as N and k grow. The Poisson distribution describes the random arrival time of events in a time interval and a radioactive rate measurement is exactly the same problem. This give us a way to assess the error affecting a rate measurement which will be equal to the root mean square of the Poisson distribution, or $\sigma = \sqrt{Var(x)}$ This means that if we measure k events, the error on the measurement will be:

$$R = k \pm \sqrt{k} \quad (\text{A.8})$$

A.5 Normal Distribution

The Normal distribution

$$N(\mu, \sigma) = e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad (\text{A.9})$$

is a limiting case of the Poisson distribution in the case of large λ values.

Let's first calculate the momentum generating function (MGF) of the Poisson distribution:

$$E(e^{tk}) = \sum_k \frac{\lambda^k e^{-\lambda}}{k!} e^{tk} = e^{-\lambda} \sum_k \frac{(\lambda e^t)^k}{k!} = e^{-\lambda} e^{e^t \lambda} = e^{\lambda(e^t - 1)} \quad (\text{A.10})$$

The MGF for the Normal distribution is easily calculated:

$$E(e^{tx}) = \int N(\mu, \sigma) e^{tx} dx = e^{t\mu + \frac{1}{2}\sigma^2 t^2} \quad (\text{A.11})$$

We would like to show that the limit of the Poisson's MGF is the Normal's MGF.

First we rescale a Poisson random variable X as:

$$Z = \frac{X - \lambda}{\sqrt{\lambda}} \quad (\text{A.12})$$

In this way, we have for the new random variable: $E(Z)=0$ and $\text{Var}(Z)=1$. From properties of the MGF:

$$M_Z(t) = e^{-t\sqrt{\lambda}} M_X\left(\frac{t}{\sqrt{\lambda}}\right) \quad (\text{A.13})$$

and therefore:

$$M_Z(t) = e^{-t\sqrt{\lambda}} e^{\lambda(e^{\frac{t}{\sqrt{\lambda}}} - 1)} = e^{t^2/2 + \frac{t^3}{3!\sqrt{\lambda}} + \dots} \quad (\text{A.14})$$

The last expression tends to $e^{t^2/2}$ as $\lambda \rightarrow \infty$ which is the MGF of a Gaussian with $E(Z)=0$ and $\text{Var}(Z)=1$. Remembering from the Binomial that $\lambda = Np$, the latter limit corresponds to a large number of trials keeping the probability fixed.

A.6 Histograms

The question we would like to answer is what is the error associated to a histogram's bin. The error on the X axis is the bin width itself, but what about Y ? A bin generally contains N entries where N can be also zero. Let's suppose that a bin contains N entries: if the true value we should see is x , what is the probability of observing N ? In such a case, we can use the Poisson distribution with $\lambda = N$ and the error turns out to be \sqrt{N} .

In conclusion, histogram bins have a Poisson error of $\pm\sqrt{N}$ when the bin content is N entries.

A histogram can indeed be regarded as made with independent Poisson distributions for the number of entries in each bin.

What stated above is fine as long as we keep in mind what does the error mean.

Since it comes from the Poisson distribution, it does not have the 68% coverage of the Normal distribution (68% is the area between $\pm\sigma$ in a Normal distribution). The Poisson distribution for small counts is also asymmetric and therefore also the error bars should be asymmetric.

Moreover, we do not know what the variance really is, since N is just an estimate. Other problems are connected to bins with $N=0$ which will have no error and this is suspicious.

Indeed a good solution to the histogram error bars cannot really exist and it would be better to draw error bars or contours around the model curve we are comparing with.

All the mentioned problems fade away as N becomes big enough to have the Poisson distribution approaching a Normal one but still one can object that if we observe N counts this is it: there is no error involved in that!

A.7 Indirect Measurements

Very often we are interested in quantities which are derived from combinations of other quantities affected by a measurement error. For example, for obtaining the area of a square we can just measure a side and then square it. The side measurement will have an error: what about the error of the square? The general problem is to derive the error σ_y of a quantity y which depends on other quantities x_1, x_2, x_3, \dots or: $y = y(x_1, x_2, x_3, \dots)$. The quantities x_i have also an error σ_{x_i} associated with them. Expanding the definition of root mean square around the average measurements (x is now the collective vector of variables):

$$\sigma_y^2 = \frac{1}{N-1} \sum_i (f_i(x) - f(\bar{x}))^2 \approx \quad (\text{A.15})$$

$$\approx \frac{1}{N-1} \sum_i \left[(x_{i,1} - \bar{x})^2 \left(\frac{\partial f_i(x)}{\partial x_1} \right)^2 + (x_{i,2} - \bar{x})^2 \left(\frac{\partial f_i(x)}{\partial x_2} \right)^2 + \dots \right] = \quad (\text{A.16})$$

$$= \left(\frac{\partial f(x)}{\partial x_1} \right)^2 \sigma_{x_1}^2 + \left(\frac{\partial f(x)}{\partial x_2} \right)^2 \sigma_{x_2}^2 + \dots \quad (\text{A.17})$$

The final formula for propagating the errors from the measured quantities x_i to the derived quantity y is:

$$\sigma_y = \sqrt{\left(\frac{\partial f(x)}{\partial x_1} \right)^2 \sigma_{x_1}^2 + \left(\frac{\partial f(x)}{\partial x_2} \right)^2 \sigma_{x_2}^2 + \dots} \quad (\text{A.18})$$

Appendix B

Basic Introduction to Quantum Mechanics

B.1 Introduction

At the turn of the century between 800s and 900s, experiments measuring the radiation spectrum from a black body were showing surprising results which were unexplained by the available theories of classical mechanics, thermodynamics and electromagnetism. The first successful equation describing the black body spectrum was due to Planck, who was forced to introduce a new constant (h , now known as Planck's constant) and to assume that the oscillators emitting radiation in the black body were "quantized" with energy $E = h\nu$. In another fundamental contribution, A.Einstein explained the photoelectric effect with the same hypothesis. In 1924, L.deBroglie proposed that matter was behaving like a wave with wavelength inversely proportional to the momentum of the particle:

$$\lambda = \frac{h}{p} \tag{B.1}$$

The hypothesis was spectacularly confirmed three years later with experiments showing the diffraction of electrons, a phenomenon known only for waves. Many facts and theoretical results were pointing towards the need of a new kind of physics for describing the microscopic world. The resulting new theory was called quantum mechanics and it had many contributors. In the following we will briefly introduce the quantum mechanics picture first developed in 1926 by E. Schrödinger, which became quickly popular for its success and similarity with older physics techniques. The quantum mechanics version developed by W.Heisenberg in 1925 was more abstract and will be more fruitful in the future in connection with relativistic formulations. The Schrödinger picture was called "wave mechanics" while the Heisenberg picture was called "matrix mechanics". Schrödinger was quickly able to prove that his wave mechanics was contained in the matrix mechanics of Heisenberg, but not the opposite case. Only in 1929, J.von Neumann was able to give a rigorous formal proof of the equivalence of the two methods.

After this short historical overview, let's concentrate on the Schrödinger picture.

First of all, the Schrödinger equation cannot be derived. Anyway, also Newton's equation $F=ma$ cannot be derived: it is a generalization of experimental facts. The Schrödinger equation was constructed in the same way: we are looking for an equation able to accommodate the known facts: after writing it, we can use it to make predictions and see if it works.

B.2 The Schrödinger Equation

From deBroglie's hypothesis, we know that matter should behave like a wave, so let's consider the simplest wave possible, a plane wave:

$$\psi = \psi_0 e^{i(\bar{k}\bar{x} - \omega t)} \quad (\text{B.2})$$

Introducing the new "quantum" relations discovered by Planck, Einstein and deBroglie: $p = \hbar k = h/\lambda, E = \hbar\omega = h\nu$ we can rewrite the plane wave as:

$$\psi = \psi_0 e^{\frac{i}{\hbar}(\bar{p}\bar{x} - E \cdot t)} \quad (\text{B.3})$$

We notice that applying specific operators to ψ we can extract important physics quantities like energy and momentum:

$$\nabla\psi = \frac{i}{\hbar}p\psi \quad (\text{B.4})$$

$$\frac{\partial}{\partial t}\psi = -\frac{i}{\hbar}E\psi \quad (\text{B.5})$$

Rearranging the terms from the last equations we obtain the differential operators acting on the wave function ψ and extracting the energy and momentum from it:

$$\hat{p}\psi = -i\hbar\nabla\psi = p\psi \quad (\text{B.6})$$

$$\hat{E}\psi = i\hbar\frac{\partial}{\partial t}\psi = E\psi \quad (\text{B.7})$$

In the latter equations we introduced the "hat" notation which denotes operators. For example, \hat{E} is the total energy operator, not the scalar number E denoting the classical total energy.

Now we can rewrite Eq. B.7 as

$$i\hbar\frac{\partial}{\partial t}\psi = (\hat{T} + \hat{V})\psi \quad (\text{B.8})$$

where we used $\hat{E} = \hat{T} + \hat{V}$. We know the operator extracting the total energy E . We need the operator for the kinetic energy T . Classically, the kinetic energy can be rewritten as $T = \frac{p^2}{2m}$ where p is the momentum and m the mass of the particle. We know already the momentum operator from Eq. B.6, so we can rewrite the kinetic energy operator \hat{T} as:

$$\hat{T} = \frac{\hat{p} \cdot \hat{p}}{2m} = -\frac{\hbar^2}{2m}\nabla^2 \quad (\text{B.9})$$

Putting together Eq. B.8 and Eq. B.9 we obtain the celebrated **Schrödinger Equation**:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + \hat{V} \psi \quad (\text{B.10})$$

The last equation is a **linear differential equation** with a **first derivative with respect to time and second derivative with respect to space**. The Maxwell and other classical wave equations had a second derivative in respect to time. Moreover, the Schrödinger equation is **complex**. Therefore, the Schrödinger equation looks like a complex version of the heat/diffusion equation.

B.3 Observables

The solution of the Schrödinger Equation is a wavefunction ψ which could be complex, while our measurements in the laboratory are real numbers.

The connection to measurements is given by the **Born rule** (1926): every observable (position, momentum, etc..) is associated to an **hermitian operator** \hat{A} . Let restrict us only to discrete cases. The operator \hat{A} will have a discrete spectrum of eigenvalues a_i and eigenfunctions f_i when acting on the wavefunction ψ . We can therefore expand the wavefunction on the eigenfunction's basis:

$$\psi = c_1 f_1 + c_2 f_2 + \dots = \sum_i c_i f_i \quad (\text{B.11})$$

Every function f_i corresponds to a specific eigenvalue a_i and the coefficients c_i are in general complex numbers.

Now the core concept of the Born's rule: the **probability** to make a measurement and find a result equal to the eigenvalue a_x is given by:

$$P(a_x) = \frac{|\langle f_x | \psi \rangle|^2}{|\langle \psi | \psi \rangle|^2} = \frac{|c_x|^2}{\sum_k |c_k|^2} \quad (\text{B.12})$$

Usually, we assume that the wavefunctions are normalized to 1 (as we do in probability theory): $|\langle \psi | \psi \rangle|^2 = 1$. The general notation $\langle f | g \rangle$ denotes an integral over a certain variable. In the space basis the wavefunctions are functions of time and space and all the integrals are done with respect to x and the normalization condition will be (in one dimension):

$$|\langle \psi | \psi \rangle|^2 = \int_{-\infty}^{+\infty} |\psi|^2 dx = \int_{-\infty}^{+\infty} \psi \psi^* dx \quad (\text{B.13})$$

Now we have a recipe for calculating observables in quantum mechanics. Actually what we can do is calculating probabilities for the outcome of an experiment.

B.4 Commutation Relations

Real numbers commute with respect to the multiplication operation but it might not be the case for two operators \hat{A} and \hat{B} applied to a wavefunction ψ .

$$\hat{A}\hat{B}\psi \neq \hat{B}\hat{A}\psi \Rightarrow (\hat{A}\hat{B} - \hat{B}\hat{A})\psi \neq 0 \Rightarrow [\hat{A}, \hat{B}]\psi \neq 0 \quad (\text{B.14})$$

It is for example the case of the space and momentum operators. Neglecting the wavefunction ψ which we always assume the operator expressions are applied to:

$$[\hat{x}, \hat{p}] = i\hbar \quad (\text{B.15})$$

The fact that in quantum mechanics we have non-commuting observables will have very important consequences as we shall see in the next section.

B.5 The Heisenberg Uncertainty Principle

The Heisenberg uncertainty principle states in its simpler form that we cannot measure position and momentum of a particle with arbitrary accuracy. More generally, it states that we cannot be arbitrarily precise in measuring at the same time two non-commuting observables, like space and momentum, for example.

Actually, the uncertainty principle is a theorem which can be derived from the non-commutativity of observables.

Let's consider two non commuting operators:

$$[\hat{A}, \hat{B}] = i\hat{C} \quad (\text{B.16})$$

The commutator gives a third operator \hat{C} where we extracted the imaginary factor for convenience. Now we define (again in analogy with probability theory) the average of an observable considering for the sake of concreteness the space basis:

$$\langle \hat{A} \rangle = \int \psi^* \hat{A} \psi \quad (\text{B.17})$$

We can now define the displacement of an operator from its average:

$$\Delta \hat{A} = \hat{A} - \langle \hat{A} \rangle \quad (\text{B.18})$$

It is easy to verify that the commutation relation $[\Delta \hat{A}, \Delta \hat{B}] = i\hat{C}$ still holds. Now we introduce the function $I(\alpha)$ where α is a real non-zero number:

$$I(\alpha) = \int |\alpha \Delta \hat{A} - i \Delta \hat{B}|^2 dx \quad (\text{B.19})$$

The latter function value is either zero or positive: $I(\alpha) \geq 0$.

Let's work out the integrand of I:

$$|\alpha \Delta \hat{A} - i \Delta \hat{B}|^2 = (\alpha \Delta \hat{A} - i \Delta \hat{B})(\alpha \Delta \hat{A} + i \Delta \hat{B}) = \alpha^2 \Delta \hat{A}^2 + i \alpha [\Delta \hat{A}, \Delta \hat{B}] - \Delta \hat{B}^2 \quad (\text{B.20})$$

Taking the integral of the last expression, remembering the definition of average of an operator and that $I(\alpha) \geq 0$ we obtain:

$$\alpha^2 \langle \Delta \hat{A}^2 \rangle + \alpha \langle \Delta \hat{C} \rangle^2 - \langle \Delta \hat{B}^2 \rangle \geq 0 \quad (\text{B.21})$$

Solving the last second order equation for α we realize that it is satisfied for every α only if:

$$\langle \Delta \hat{A}^2 \rangle \cdot \langle \Delta \hat{B}^2 \rangle \geq \frac{1}{4} \langle \Delta \hat{C} \rangle^2 \quad (\text{B.22})$$

The last equation represents the **Heisenberg uncertainty principle** and the average of the square of the displacement from the mean (eg: $\langle \Delta \hat{A}^2 \rangle$) expresses the known statistical notion of **root mean square**. Applying the result to space and momentum:

$$\langle \Delta \hat{x}^2 \rangle \cdot \langle \Delta \hat{p}^2 \rangle \geq \frac{1}{4} \hbar^2 \quad (\text{B.23})$$

B.6 Stationary Schrödinger Equation

In many applications, the potential V does not depend on time. If this is true, we can factorize the wavefunction Ψ in one part depending only from space and another one depending on time:

$$\Psi = \psi(x)f(t) \quad (\text{B.24})$$

Substituting the latter decomposition into the Schrödinger equation B.10 and bringing to opposite sides the x and t dependencies:

$$i\hbar \frac{\partial f}{\partial t} \frac{1}{f} = -\frac{\hbar^2}{2m} \frac{1}{\psi} \nabla^2 \psi \quad (\text{B.25})$$

If both sides are equal, then both sides should be equal to the same number which we call E for now. We obtain two equations:

$$i\hbar \frac{\partial f}{\partial t} \frac{1}{f} = E \quad (\text{B.26})$$

$$-\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi = E\psi \quad (\text{B.27})$$

The first one is easily solved: $f(t) = e^{-i\frac{E}{\hbar}t}$. From dimensional analysis, it is clear that the constant E is an energy. Remembering the definition of the total energy operator \hat{E} which we call here \hat{H} (the ‘‘hamiltonian’’) we can rewrite the second equation as:

$$\hat{H}\psi = E\psi \quad (\text{B.28})$$

The last equation is an eigenvalue equation and its solution will provide the eigenfunctions ψ along with the eigenvalues E .

The full solution reads:

$$\Psi(x, t) = \psi(x)e^{-i\frac{E}{\hbar}Et} \quad (\text{B.29})$$

The remarkable fact is that:

$$\Psi\Psi^* = \psi(x)e^{-i\frac{\hbar}{\hbar}Et}\psi(x)e^{+i\frac{\hbar}{\hbar}Et} = \psi(x)^2 \quad (\text{B.30})$$

does not depend on time. This means that our probability measure does not depend on time. Eq. B.28 is commonly called **stationary Schrödinger equation**.

B.7 Application: the Square Well

Applying the Schrödinger equation to eg the Hydrogen atom is a quite complex task, although Schrödinger himself calculated it rather early after the discovery of the equation finding nice agreement with the known experimental results and with Bohr's previous calculations. Let's apply the equation to one of the simplest cases we can think of (besides the free particle case, where $V=0$): a square well potential in one single dimension for just one particle. The square well potential is constructed such that $V(x)=0$ for $0 < x < L$ with $L > 0$ a real number and $V(x) = +\infty$ otherwise.

The one dimensional Schrödinger equation will be:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi + \hat{V}\psi \quad (\text{B.31})$$

There are no forces acting on the particle inside the box with infinitely high sides, so we can use as an ansatz for the wavefunction a simple superposition of sines and cosines:

$$\psi(x, t) = [a \sin(kx) + b \cos(kx)] e^{-i\omega t} \quad (\text{B.32})$$

where a and b are in general complex numbers and (k, ω) were related to impulse and energy respectively through the deBroglie wave-particle relation. The form of the solution comes from the stationary equation derived before, since V does not depend on time.

The probability of finding a particle somewhere is given by the square of the wavefunction (Born's rule). Since the potential walls are infinite, it is impossible to find a particle at $x < 0$ and $x > L$: the wavefunction must vanish in those regions. The wavefunction should be also smooth in the sense that it cannot jump from one value inside the box to zero outside. The latter consideration forces us to choose $b=0$.

The set of eigenfunctions solving the equation Eq. B.28 must be of the form:

$$\psi_n(x, t) = a \sin(k_n x) e^{-i\omega t} \quad (\text{B.33})$$

for $0 < x < L$ and $\psi = 0$ otherwise.

Substituting in the stationary equation we obtain:

$$E_n = \frac{k^2 \hbar^2}{2m} \quad (\text{B.34})$$

which is the energy of a free particle (which is correct, inside the walls).

In order to have $\psi = 0$ on the walls, the wave numbers can only assume the values

$k_n = n\pi/L$. Substituting into Eq. B.34 we find an expression for the energy levels of a particle in an infinitely high box:

$$E_n = \frac{n^2 h^2}{8mL^2} \quad (\text{B.35})$$

Appendix C

Topics related to Astrophysics

C.1 Gravitational Binding Energy

We would like to calculate the binding energy due to the gravitational attraction of matter particles. Physically the binding energy would be equivalent to the energy needed to pull all the particles away to infinity from the initial body. The binding energy is equal to the negative of the total gravitational potential energy. For simplicity, let's consider a sphere with constant mass density.

In general, the gravitational potential V of a mass m_1 is given by

$$V(r) = -\frac{Gm_1}{r} \quad (\text{C.1})$$

with G the universal gravitational constant. The potential energy of a second mass m_2 in the potential generated by m_1 is given by:

$$U(r) = m_2V(r) = -\frac{Gm_1m_2}{r} \quad (\text{C.2})$$

Let's calculate the potential energy of a thin "crust" of matter of mass dm around a spherical distribution with constant density ρ and mass $m_{core} = (4/3)\pi r^3\rho$. The energy of the shell is:

$$dU_{shell}(r) = V(r)dm = \frac{-Gm_{core}m_{shell}}{r} = -\frac{G\left(\frac{4}{3}\pi r^3\rho\right)(4\pi r^2 dr\rho)}{r} \quad (\text{C.3})$$

After some simplifications and substituting the constant density $\rho = M/V_{sphere} = M/(4/3)\pi R^3$ we obtain:

$$dU_{shell}(r) = -G\frac{16\pi^2}{3}\rho^2 r^4 dr \quad (\text{C.4})$$

Integrating up to a radius R :

$$BE_{grav} = -\int_0^R dU = \frac{3}{5}\frac{GM^2}{R} \quad (\text{C.5})$$

Appendix D

Relativistic Dynamics and Particle Accelerators

D.1 Introduction

In the following, we will derive useful expressions for charged particles moving in electromagnetic fields relevant for the analysis of beam dynamics in circular particle accelerators. Knowledge of lagrangian and hamiltonian mechanics is assumed.

D.2 Relativistic Lagrangian and Hamiltonian Kinematics

In a reference frame where we are at rest, a moving clock travels a distance $dl = \sqrt{dx^2 + dy^2 + dz^2}$ in a time dt . In the reference frame attached to the clock, the traveled distance is $ds' = 0$. We ask now the question: what is the time interval dt' in the clock's reference frame? Since the metric $ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$ is invariant in all inertial reference frames:

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 = c^2 dt'^2 \quad (\text{D.1})$$

Solving for dt' :

$$dt' = dt \sqrt{1 - \frac{dx^2 + dy^2 + dz^2}{c^2 dt^2}} = dt \sqrt{1 - \frac{v^2}{c^2}} = \frac{dt}{\gamma} = \frac{ds}{c} \quad (\text{D.2})$$

where we introduced the factor $\gamma = 1/\sqrt{1 - v^2/c^2}$.

Let us now construct an action S_{rel} for relativistic dynamics. Lorentz invariance requires S_{rel} to be a scalar and the only available scalar is ds (it is the same in every inertial reference frame) or something proportional to it:

$$S_{rel} = -\alpha \int ds = \alpha \int \frac{ds}{c} c = -\alpha \int c \sqrt{1 - \frac{v^2}{c^2}} \quad (\text{D.3})$$

where α is a positive constant and the “-” sign will turn out to be a convenient choice later. Since by definition $S = \int \mathcal{L} dt$ with \mathcal{L} the Lagrangian, we have:

$$\mathcal{L} = -\alpha c \sqrt{1 - \frac{v^2}{c^2}} \quad (\text{D.4})$$

The constant α can be fixed considering that the non-relativistic limit should match the classical expression $\mathcal{L}_{class} = \frac{1}{2}mv^2$. Considering low velocities, we can expand the Lagrangian in the small parameter v/c :

$$\mathcal{L} \approx -\alpha c + \frac{\alpha v^2}{2c} + \mathcal{O}(v^2/c^2) \quad (\text{D.5})$$

Comparing the last expression with \mathcal{L}_{class} we have finally:

$$\mathcal{L} = -mc^2 \sqrt{1 - \frac{v^2}{c^2}} \quad (\text{D.6})$$

Having the Lagrangian, we can calculate the momentum (let’s forget for notational simplicity that momenta and velocities are vectors):

$$p = \frac{\partial \mathcal{L}}{\partial v} = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (\text{D.7})$$

Having p , we can directly derive the hamiltonian via the Legendre transformation $\mathcal{H} = pv - \mathcal{L}$ obtaining:

$$\mathcal{H} = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (\text{D.8})$$

As a check, we can again expand the last expression in the limit of small velocities in the parameter v/c :

$$\mathcal{H} \approx mc^2 + \frac{1}{2}mv^2 + \mathcal{O}(v^2/c^2) \quad (\text{D.9})$$

The last formula agrees with the classical expression for the kinetic energy $E_{kin} = \frac{1}{2}mv^2$ but the additional relativistic term mc^2 indicates that also a body at rest has a non-zero energy $E = mc^2$ directly proportional to its mass.

The last expressions still contains the velocity v , while the Hamiltonian should be a function of positions and momenta only (or other two canonically conjugated variables). Considering p^2 and substituting it in \mathcal{H}^2 , v can be eliminated obtaining finally:

$$\mathcal{H} = c\sqrt{p^2 + m^2c^2} \quad (\text{D.10})$$

Since in this case the Hamiltonian is equivalent to the total energy E , we can rewrite the last equation also in the useful and common expression:

$$E^2 - c^2p^2 = m^2c^4 \quad (\text{D.11})$$

which shows the 4-vector character of energy and impulse. In fact, if we arrange E (scalar) and p (3-vector) in a 4-vector $(E/c, p)$, Eq. D.11 corresponds to the square

of it. Since the square of a 4-vector (its norm) is the same in every inertial reference frame, Eq. D.11 holds in all of them and therefor its usefulness. We note also the similarity between the 4-vectors $(E/c, p)$ and (ct, x) : the energy plays the role of time component, while the momentum plays the role of space component. Another correspondence is that indeed the couples (energy,time) and (momentum,space) are canonically conjugated variables.

D.3 Relativistic Dynamics with Electromagnetic Fields

Let's add to the action S we derived before, another scalar containing the electromagnetic potentials arranged into the 4-vector (ϕ, A_i) (roman indices run from 1 to 3, greek indices from 1 to 4):

$$S_{EM} = -\frac{e}{c} \int A_\mu dx^\mu \quad (\text{D.12})$$

Considering $S = S_{rel} + S_{EM}$ and decomposing the index sum:

$$S = \int \left[-mc^2 \sqrt{1 - \frac{v^2}{c^2}} dt + \frac{e}{c} A_i dx^i - e\phi dt \right] = \int \left[-mc^2 \sqrt{1 - \frac{v^2}{c^2}} + \frac{e}{c} A_i v^i - e\phi \right] dt \quad (\text{D.13})$$

Therefore, the relativistic Lagrangian is:

$$\mathcal{L} = -mc^2 \sqrt{1 - \frac{v^2}{c^2}} + \frac{e}{c} A_i v^i - e\phi \quad (\text{D.14})$$

As we did before, we can calculate the momentum and then the Hamiltonian obtaining:

$$p_i = mv_i \gamma + \frac{e}{c} A_i \quad (\text{D.15})$$

and

$$\mathcal{H} = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} + e\phi \quad (\text{D.16})$$

Again, we have to eliminate the velocity. A faster way to derive the Hamiltonian is to consider Eq. D.11 and observing that in the electromagnetic case we have the substitutions: $p \rightarrow p + \frac{e}{c} A$ and $H \rightarrow H - e\phi$. The last consideration permits us to write right away:

$$\left(\frac{\mathcal{H} - e\phi}{c} \right)^2 - c^2 \left(p - \frac{e}{c} A \right)^2 = m^2 c^4 \quad (\text{D.17})$$

and finally:

$$\mathcal{H} = \sqrt{m^2 c^4 + c^2 \left(p - \frac{e}{c} A \right)^2} + e\phi \quad (\text{D.18})$$

D.4 Motion on a curve: Frenet Coordinates

For parameterizing the trajectory of a particle, few concepts from differential geometry are useful.

Let us consider a function in E_3 space $P : (a, b) \rightarrow E_3$ with (a, b) an open set of \mathcal{R} . The function P is *regular* if $P \in C^1$ and $\frac{\partial P}{\partial \epsilon} \neq 0 \forall \epsilon \in (a, b)$.

If we introduce a coordinate system (x, y, z) in E_3 , a representation of P in such coordinates is $P(\epsilon) = (x = x(\epsilon), y = y(\epsilon), z = z(\epsilon))$. All the last 3 functions must be C^1 and their derivatives must never be zero at the same time.

A subset Γ of E_3 is a *regular curve* if Γ may be regarded as the image of a regular function $P : I \rightarrow E_3$. Regular curves can have many representations and to be regular, they must be regular at least in one representation.

Let's now introduce the curvilinear abscissa (or curvilinear coordinate). Let Γ be a regular curve, its curvilinear coordinate is:

$$S(\xi) - S(\xi_0) = \int_{\xi_0}^{\xi} \left\| \frac{dP}{d\xi} \right\| d\xi = \int_{\xi_0}^{\xi} \sqrt{\left(\frac{dx}{d\xi} \right)^2 + \left(\frac{dy}{d\xi} \right)^2 + \left(\frac{dz}{d\xi} \right)^2} d\xi \quad (\text{D.19})$$

S has a metric meaning which the previous parameter ξ did not have: S really measures the amount of space we travelled on the curve.

At every point of the curve parameterized by S , we can attach three orthonormal vectors defined as follows:

$$t = \frac{dP}{dS} \quad (\text{D.20})$$

$$n = \frac{dt}{dS} / \left\| \frac{dt}{dS} \right\| \quad (\text{D.21})$$

$$b = t \wedge n \quad (\text{D.22})$$

The vector t is called *tangent vector*, n is the *normal vector* and b is the *binormal vector*. These three vectors have unitary norm and they are orthogonal to each other.

Let's verify for example that $t \perp n$. Consider $t(s) \cdot t(s) = 1$, then taking the d/ds derivative of both sides, $2t(s) \frac{dt}{ds} = 0$ which implies $t \perp \frac{dt}{ds}$.

The three defined vectors are "intrinsic" in the sense that assigned the curve, also the vectors are fixed. As the parameter S changes, the three vectors "rotate" along the curve. The **Frenet Formulas** express how the vectors change along the curve. Let's see how first considering how the binormal vector changes:

$$\frac{db(s)}{ds} = \frac{d}{ds} (t(s) \wedge n(s)) = \frac{dt}{ds} \wedge n + t \wedge \frac{dn}{ds} = cn \wedge n + t \wedge \frac{dn}{ds} = t \wedge \frac{dn}{ds} \quad (\text{D.23})$$

In the last equation, we used the fact that dt/ds is in the direction of n and we have introduced a proportionality constant c which we call *curvature*. Then we used the fact that $n \wedge n = 0$. The result informs us that $db/ds \perp b$ and therefore db/ds is in the direction of n :

$$\frac{db}{ds} = \tau n \quad (\text{D.24})$$

The proportionality constant τ is called *torsion*.

Now since the vectors t, n, b are orthogonal, $n = b \wedge t$. Taking the d/ds derivative of the last relation:

$$\frac{dn}{ds} = \frac{db}{ds} \wedge t + b \wedge \frac{dt}{ds} = \tau u \wedge t + b \wedge cn = -\tau b - ct \quad (\text{D.25})$$

We can summarize the relations obtained up to now in the following system (the Frenet Formulas):

$$\frac{dt}{ds} = cn \quad (\text{D.26})$$

$$\frac{dn}{ds} = -ct - \tau b \quad (\text{D.27})$$

$$\frac{db}{ds} = \tau n \quad (\text{D.28})$$

$$(\text{D.29})$$

or in matrix form:

$$\frac{d}{ds} \begin{pmatrix} t \\ n \\ b \end{pmatrix} = \begin{pmatrix} 0 & c & 0 \\ -c & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix} \cdot \begin{pmatrix} t \\ n \\ b \end{pmatrix} \quad (\text{D.30})$$

Note that the matrix is antisymmetric and this has an obvious connection to the generalized angular velocity. The curvature and the torsion are *invariants*, in the sense that their value depends only on the curve and not on the representation.

D.5 Frenet Coordinates in Circular Accelerators

In describing the orbit in circular accelerators, it is useful to consider instead of coordinates with respect to a certain point (eg the accelerator's geometric centre), coordinates describing the deviations from a reference orbit. Let's introduce three Frenet unit vectors $(\hat{x}, \hat{y}, \hat{s})$ attached to the orbiting particle such that $\hat{x} \wedge \hat{y} = \hat{s}$. The unit vectors \hat{x} and \hat{y} are transverse to the particle's orbit while \hat{s} is tangential to it (plays the role of the vector t of the previous section). The position of the particle is now described by

$$R(x, y, s) = x\hat{x} + y\hat{y} + R_0(s) \quad (\text{D.31})$$

where R_0 is the vector describing the reference orbit. The following relations hold:

$$\hat{s} = \frac{dR_0}{ds} \quad \frac{d\hat{x}}{ds} = \frac{1}{\rho}\hat{s} \quad \frac{d\hat{y}}{ds} = 0 \quad \frac{d\hat{s}}{ds} = -\frac{1}{\rho}\hat{x} \quad (\text{D.32})$$

The parameter ρ is the curvature, \hat{x} measures the "horizontal" displacement and \hat{y} the "vertical" one with respect to the reference trajectory. The variable s follows the reference orbit.

Our aim is to rewrite the Hamiltonian in the new coordinates. The generating function of the transformation is:

$$F_3(\bar{x}_{new}, \bar{P}_{old}, t) = \bar{P} \cdot \bar{R}(x, y, s) \quad (\text{D.33})$$

where \bar{x}_{new} are the new coordinates (x,y,s) and \bar{P}_{old} are the new momenta. With the generating function we can derive the new momenta:

$$P_x = \frac{\partial F}{\partial x} = \bar{P} \hat{x} \quad (\text{D.34})$$

$$P_y = \frac{\partial F}{\partial y} = \bar{P} \hat{y} \quad (\text{D.35})$$

$$P_s = \frac{\partial F}{\partial s} = \bar{P} \left(1 + \frac{x}{\rho}\right) \hat{s} \quad (\text{D.36})$$

Since the generating function does not depend on the time, the new Hamiltonian has the same form of the old one, but with the new canonical coordinates. Remembering the implicit form of the Hamiltonian in Eq. D.17, we note that it has the form of a 4-vector with norm mc^2 . By Lorentz invariance, the norm should stay the same in every reference frame. If we now define for convenience:

$$A_s = \left(1 + \frac{x}{\rho}\right) \bar{A} \hat{s} = h \bar{A} \hat{s} \quad (\text{D.37})$$

where $h = 1 + x/\rho$.

Substituting the square of the new momenta in the implicit Hamiltonian:

$$mc^2 = \left(\frac{E - q\phi}{c}\right)^2 - (P_x - qA_x)^2 - (P_y - qA_y)^2 - \left(\frac{P_s - qA_s}{1 + x/\rho}\right)^2 \quad (\text{D.38})$$

All the canonical variables have the same status, so we can explicitate everyone of them. An useful choice is to explicitate the P_s momentum. The Hamiltonian in this case reads:

$$H_s = -qA_s - \left(1 + \frac{x}{\rho}\right) \sqrt{-m^2c^2 - (P_x - qA_x)^2 - (P_y - qA_y)^2 + \left(\frac{E - q\phi}{c}\right)^2} \quad (\text{D.39})$$

In the last Hamiltonian, the coordinate s plays the role of time, so the canonical equations are now of the form:

$$\frac{d\bar{P}}{ds} = -\frac{\partial H}{\partial \bar{x}} \quad ; \quad \frac{d\bar{x}}{ds} = \frac{\partial H}{\partial \bar{P}} \quad (\text{D.40})$$

Now we would like to use the previous equations of motion in a simplified case but first let's recover the expression for the magnetic field in the new coordinates:

$$\bar{B} = \text{rot} \bar{A} = \begin{pmatrix} \frac{1}{h} \frac{\partial A_s}{\partial y} - \frac{\partial A_y}{\partial s} \\ \frac{\partial A_x}{\partial s} - \frac{1}{h} \frac{\partial A_s}{\partial x} \\ \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \end{pmatrix} \quad (\text{D.41})$$

As a way to check the validity of the last Hamiltonian, let's consider the special case of a magnetic field such that $A_x = A_y = 0$ and a particle moving exactly on the reference trajectory. This means that $x = y = 0$ and $P_x = P_y = 0$. Using the last conditions in the canonical equations of motion:

$$0 = \frac{dP_x}{ds} = -\frac{\partial H}{\partial x} = q\frac{\partial A_s}{\partial x} + \frac{1}{\rho}\sqrt{\frac{E^2}{c^2} - m^2c^2} \quad (\text{D.42})$$

Since the square root in the last formula is the particle's momentum on the reference trajectory P_{ref} and $B_y^{ref} = -(1/h)\partial A_s/\partial x$ we obtain:

$$B_y^{ref} \cdot \rho = \frac{1}{q}P_x^{ref} \quad (\text{D.43})$$

An analogous equation can be obtained for P_y^{ref} and we recognize the usual relationship between momentum and radius for a particle moving in a constant magnetic field. The combination $B \cdot \rho$ is known as *magnetic rigidity*. Since B and ρ may vary along the trajectory, the more general formula holds:

$$B_y^{ref}(s) \cdot \rho(s) = \frac{1}{q}P_x^{ref} \quad (\text{D.44})$$

Another special case can be considered turning off all the electromagnetic fields. In this case, the canonical equations of motion are (let's neglect y for simplicity and use the energy as Hamiltonian):

$$\begin{cases} \dot{x} = \frac{c^2 P_x}{E} \\ \dot{P} = \frac{c^2}{\rho E} \frac{P_s^2}{(1+x/\rho)^3} \end{cases} \quad (\text{D.45})$$

and

$$\begin{cases} \dot{s} = \frac{c^2}{E} \frac{P_s}{(1+x/\rho)^2} \\ \dot{P}_s = 0 \end{cases} \quad (\text{D.46})$$

Considering the non-relativistic reduction $\frac{E}{c^2} \approx m$, $P_x \approx m\dot{x}$ and $P_s = m\dot{s}(1+x/\rho)^2$:

$$\begin{cases} \ddot{x} = \frac{\dot{s}^2}{\rho}(1+x/\rho)^2 = \omega^2 R \\ \ddot{s} = \frac{-2c^2}{E} \frac{P_s}{(1+x/\rho)^3} \frac{\dot{x}}{\rho} = -2\omega\dot{x} \end{cases} \quad (\text{D.47})$$

In the last parts of the equations of motion, we used $R = x + \rho$ and $\omega = \dot{s}/\rho$. The first equation expresses the centripetal acceleration, while the second the Coriolis acceleration.

Appendix E

Nuclear Science Nobel Laureates

Year	Laureate(s)	Contribution
Chemistry		
1908	Ernest Rutherford	Radioactive Decays
1911	Marie Curie	Discovery of Polonium and Radium
1921	Frederick Soddy	Discovery of Isotopes
1922	Francis Aston	Separation of the Ne isotopes
1934	Harold Urey	Discovery of Deuterium
1935	Frederic and Irene Joliot-Curie	Artificial Radioactivity
1943	Georg Hevesy	Radiotracers and Neutron Activation Analysis and Neutron Activation Analysis
1944	Otto Hahn	Nuclear Fission
1951	Edwin McMillan, Glenn Seaborg	Discovery of Np and Pu
1960	Willard Libby	^{14}C and ^3H dating
Physics		
1901	Wilhelm Roentgen	X-rays
1903	Henri Becquerel, Marie and Pierre Curie	Radioactivity
1935	James Chadwick	Discovery of the Neutron
1938	Enrico Fermi	Search of transuranic elements, controlled nuclear fission
1939	Ernest Lawrence	Development of the Cyclotron
1951	John Cockcroft, Ernest Walton	First accelerator producing nuclear reactions
1961	Rudolf Moessbauer	Moessbauer effect
1963	M. Goeppert-Meyer, J. Jensen, E Wigner	Nuclear Structure Studies
1968	Luis Alvarez	Electron capture decay
1975	B. Mottelson, A. Bohr	Collective models of the nucleus
1983	William Fowler	Nucleosynthesis in stars

Appendix F

Periodic Table of Elements

