Astroparticle Physics

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Lecture 2

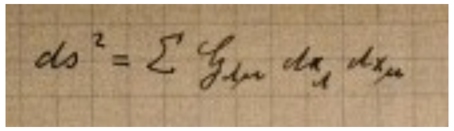
Brief History of GR

1907: While working at the Bern's Patent Office, A. Einstein realized how to start generalizing special relativity to generic reference frames.

1908: First paper about acceleration and relativity by A. Einstein. In this paper he states the Equivalence Principle and derives time dilation caused by gravitational fields.

1911: Second paper by A. Einstein on time dilation by gravitational fields where also light deviation by massive bodies was approximately derived.

1912: Einstein consults with M. Grossman about non-euclidean geometry.



October 1915: First guess: $R_{ij} = T_{ij}$

November 1915: Einstein publishes the General Theory of Relativity as we know it today. D. Hilbert obtained the same equations almost at the same time.

1919: Eddington confirms the deviation of light formula from GR using a solar eclipse in Brazil.

1959: Pound-Rebka Experiment (gravitational red shift)

1971: Hafele-Keating Experiment (time dilation)

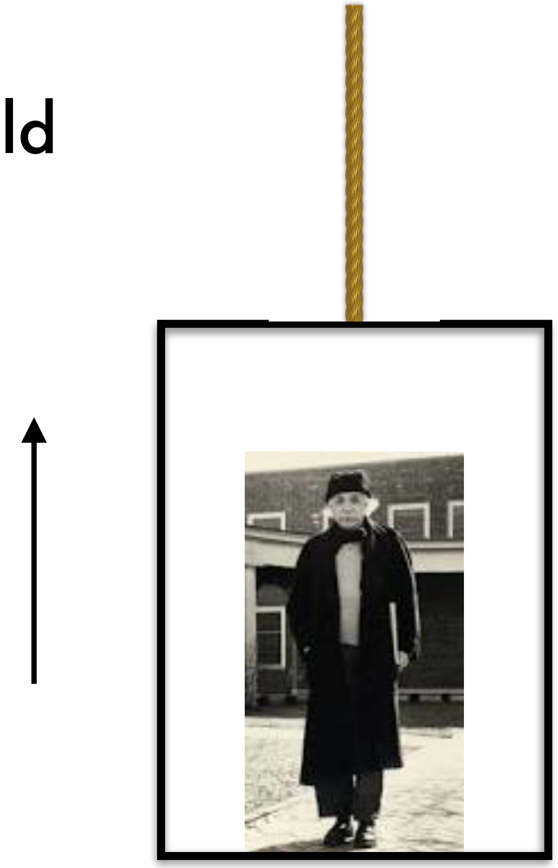
1974: Hulse-Taylor binary pulsar.

2004: Gravity Probe-B and frame dragging (published in 2011)

2016: Direct detection of gravitational waves by LIGO

The Equivalence Principle

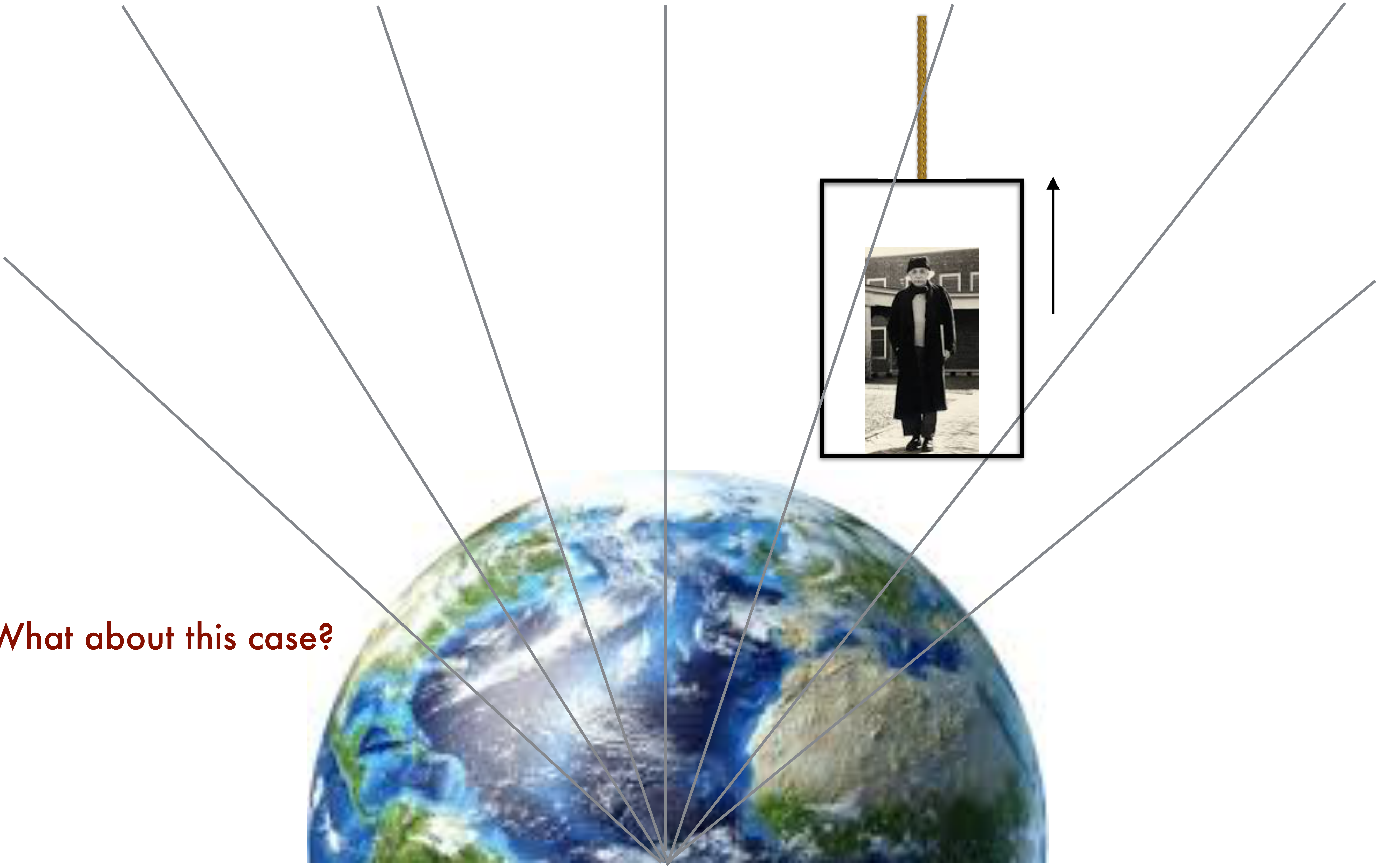
Observer in an
uniform gravitational field



Accelerated Observer

How do you tell the difference?

The Equivalence Principle

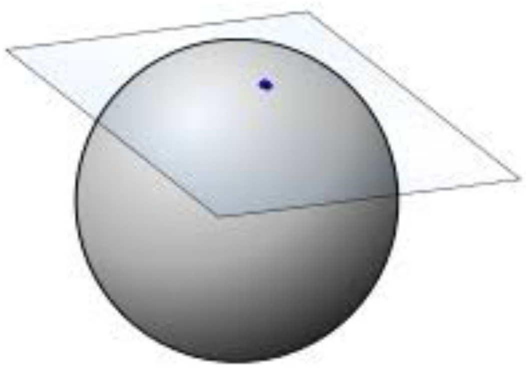


What about this case?

The Equivalence Principle

Strong Equivalence Principle:

At every space-time point in a gravitational field, it is possible to choose a locally inertial coordinate system such that in a sufficiently small neighbourhood of that point, the laws of nature can be expressed in the same form as in an unaccelerated coordinate system.



Weak Equivalence Principle:

Change "laws of nature" with "laws of motion of free-falling bodies" (gravity).

Free-Falling Bodies

$$\frac{d^2\xi^\mu}{d\tau^2} = 0$$

Equation of motion for a free-falling body

Changing to a generic coordinate system:

$$\frac{d}{d\tau}\left(\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{dx^\mu}{d\tau}\right) = \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{d^2x^\mu}{d\tau^2} + \frac{\partial^2\xi^\alpha}{\partial x^\mu \partial x^\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

Equation of motion:

$$\boxed{\frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0}$$

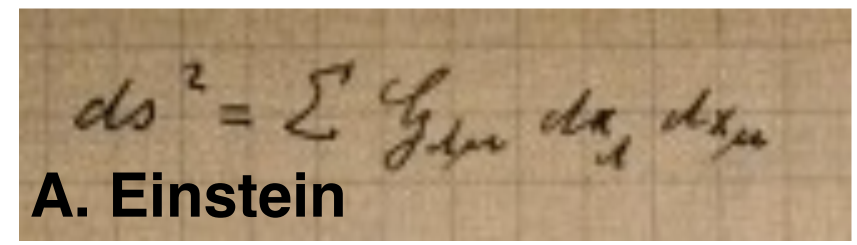
Metric and Geodetics

Metric

Gives the “distance” between two points.

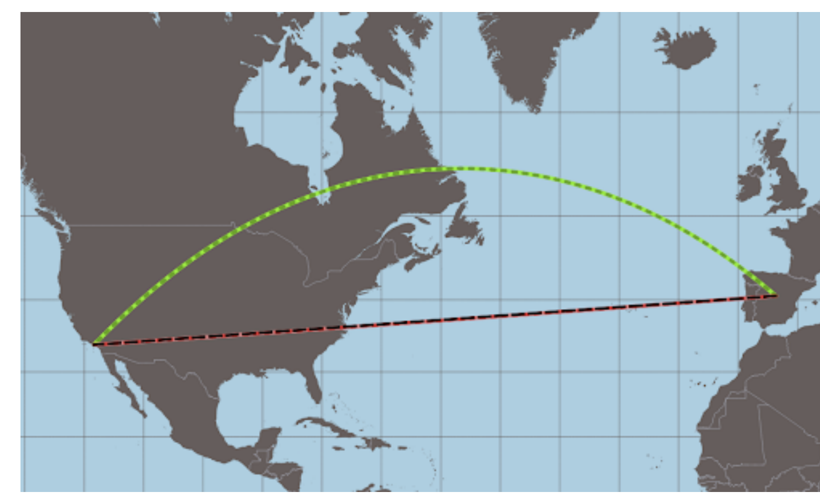
In cartesian coordinates: $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$

In a general space: $d^2 = g_{ij} x_i x_j$



Geodesic

Shortest path between two points



Geodesic equation
$$\frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0$$

Christoffel Symbols
$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \right)$$

The <u>metric</u> is the only information needed!

Einstein Equations

Die Feldgleichungen der Gravitation.

Von A. Einstein.

In zwei vor kurzem erschienenen Mitteilungen[1] habe ich gezeigt, wie man zu Feldgleichungen der Gravitation gelangen kann, die dem Postulat allgemeiner Relativität entsprechen, d. h. die in ihrer allgemeinen Fassung beliebigen Substitutionen der Raumzeitvariabeln gegenüber kovariant sind.

Der Entwicklungsgang war dabei folgender. Zunächst fand ich Gleichungen, welche die Newtonsche Theorie als Näherung enthalten und beliebigen Substitutionen von der Determinante 1 gegenüber kovariant waren. Hierauf fand ich, daß diesen Gleichungen allgemein kovariante entsprechen, falls der Skalar des Energietensors der »Materie« verschwindet. Das Koordinatensystem war dann nach der einfachen Regel zu spezialisieren, daß $\sqrt{-g}$ zu 1 gemacht wird, wodurch die Gleichungen der Theorie eine eminente Vereinfachung erfahren. Dabei mußte aber, wie erwähnt, die Hypothese eingeführt werden, daß der Skalar des Energietensors der Materie verschwinde.

Neuerdings finde ich nun, daß man ohne Hypothese über den Energietensor der Materie auskommen kann, wenn man den Energietensor der Materie in etwas anderer Weise in die Feldgleichungen einsetzt, als dies in meinen beiden früheren Mitteilungen geschehen ist. Die Feldgleichungen für das Vakuum, auf welche ich die Erklärung der Perihelbewegung des Merkur gegründet habe, bleiben von dieser Modifikation unberührt. Ich gebe hier nochmals die ganze Betrachtung, damit der Leser nicht genötigt ist, die früheren Mitteilungen unausgesetzt heranzuziehen.

Aus der bekannten Riemannschen Kovariante vierten Ranges leitet man folgende Kovariante zweiten Ranges ab:

$$G_{im} = R_{im} + S_{im} \tag{1}$$

$$R_{im} = -\sum_{l} \frac{\partial \begin{Bmatrix} im \\ l \end{Bmatrix}}{\partial x_l} + \sum_{l\varrho} \begin{Bmatrix} il \\ \varrho \end{Bmatrix} \begin{Bmatrix} m\varrho \\ l \end{Bmatrix} \tag{1a}$$

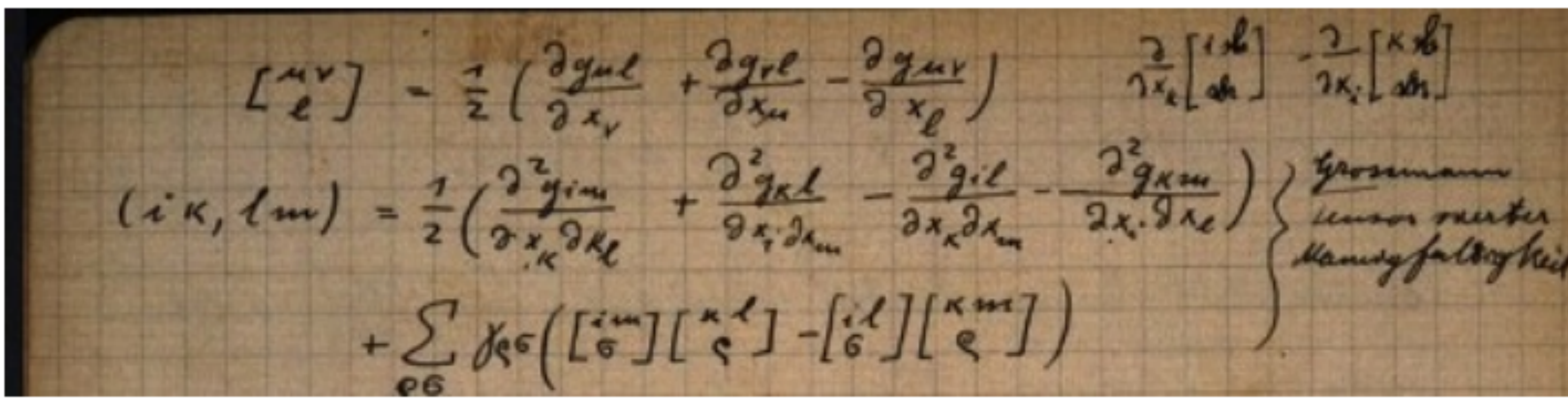
$$S_{im} = \sum_{l} \frac{\partial \begin{Bmatrix} il \\ l \end{Bmatrix}}{\partial x_m} - \sum_{l\varrho} \begin{Bmatrix} im \\ \varrho \end{Bmatrix} \begin{Bmatrix} \varrho l \\ l \end{Bmatrix} \tag{1b}$$

[1] Sitzungsber. XLIV, S. 778 und XLVI, S. 799, 1915.

Einstein Equations

Requirements:

1) It should be a tensor equation (the same in any coordinate system).
2) In analogy to other situations known in physics, it should be of second order at most in the relevant variable (the gravitational potential, or in this case the metric tensor).
3) The equation must reduce to the Poisson equation in the non-relativistic limit.
4) The source of the gravitational field should be the energy-momentum tensor T.
5) If T=0, space-time must be flat



First time the Riemann tensor appears in Einstein's notebooks.

Einstein Equations

$$R_{\alpha\beta} - g_{\alpha\beta}R = T_{\alpha\beta}$$

The Ricci tensor depends from the metric and its derivatives. Again, the metric is all what we need (together with T) for calculating the equations.

Einstein equations are non-linear! Unlike the Maxwell equations case, specifying the sources is not sufficient for calculating the fields. In this case, the energy-momentum tensor determines the geometry and the geometry modify the energy momentum tensor: very complex problem in the general case.

Properties:

1) It is a tensor equation (the same in any coordinate system).
2) In analogy to other situations known in physics, it is of second order at most in the relevant variable (the gravitational potential, or in this case the metric tensor).
3) The equation must reduce to Newtonian gravity in the non-relativistic limit.
4) The source of the gravitational field is the energy-momentum tensor T.

The Cosmological Principle

<u>*The Universe is spatially homogeneous and isotropic*</u>

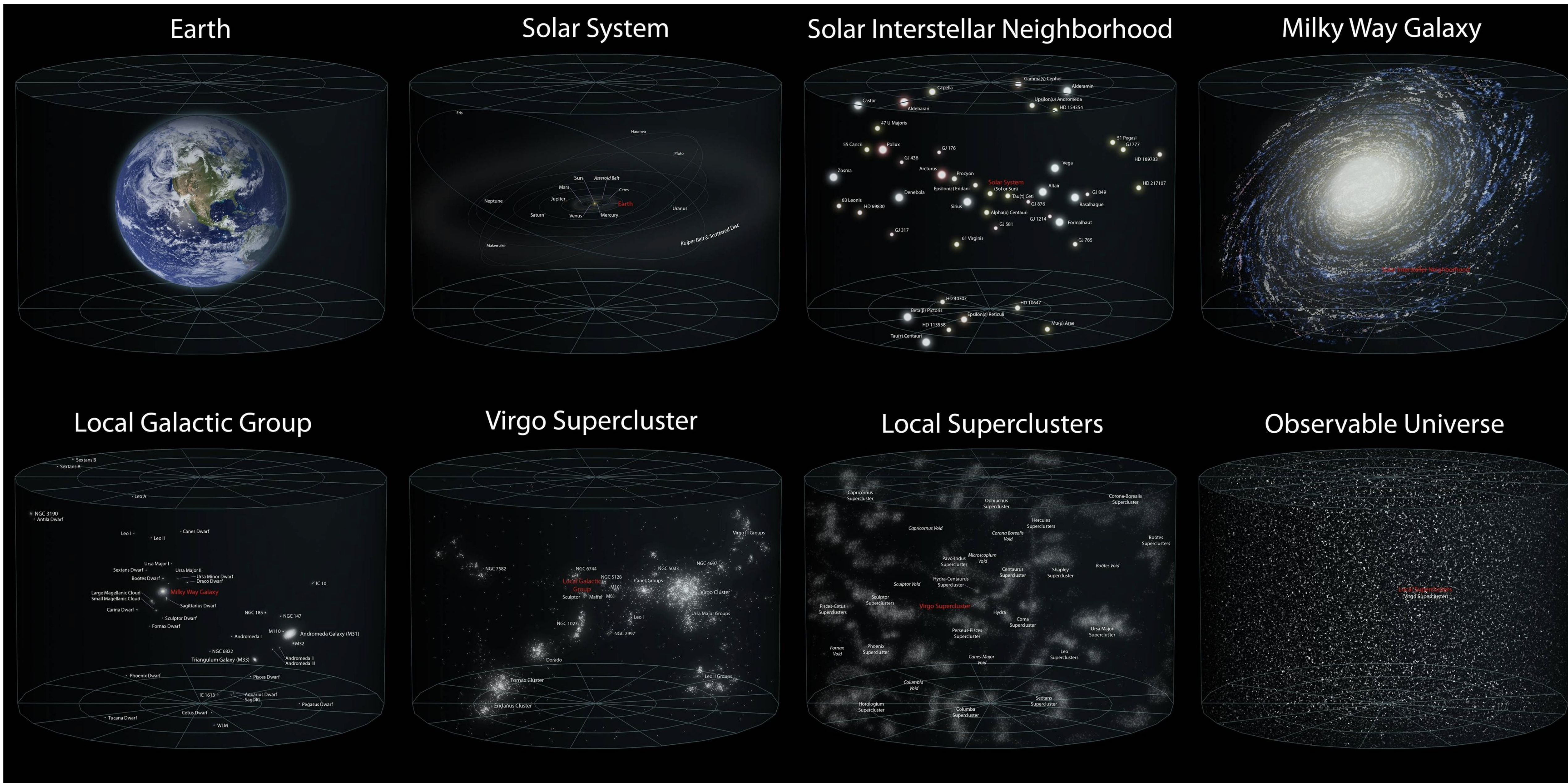


Image from Cryhavoc

The FLRW Metric

From the Cosmological Principle

we obtain the Friedmann-Lemaitre-Robertson-Walker metric:

$$\boxed{ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right]}$$

Noting the rescaling invariance

$$R \to \frac{R}{\lambda}$$
$$r \to \lambda r$$
$$k \to \frac{k}{\lambda^2}$$

we can always choose k=+1,0,-1.
R can be scaled to e.g. 1 for t=today. Usually R(t) is called a(t): the scale factor.

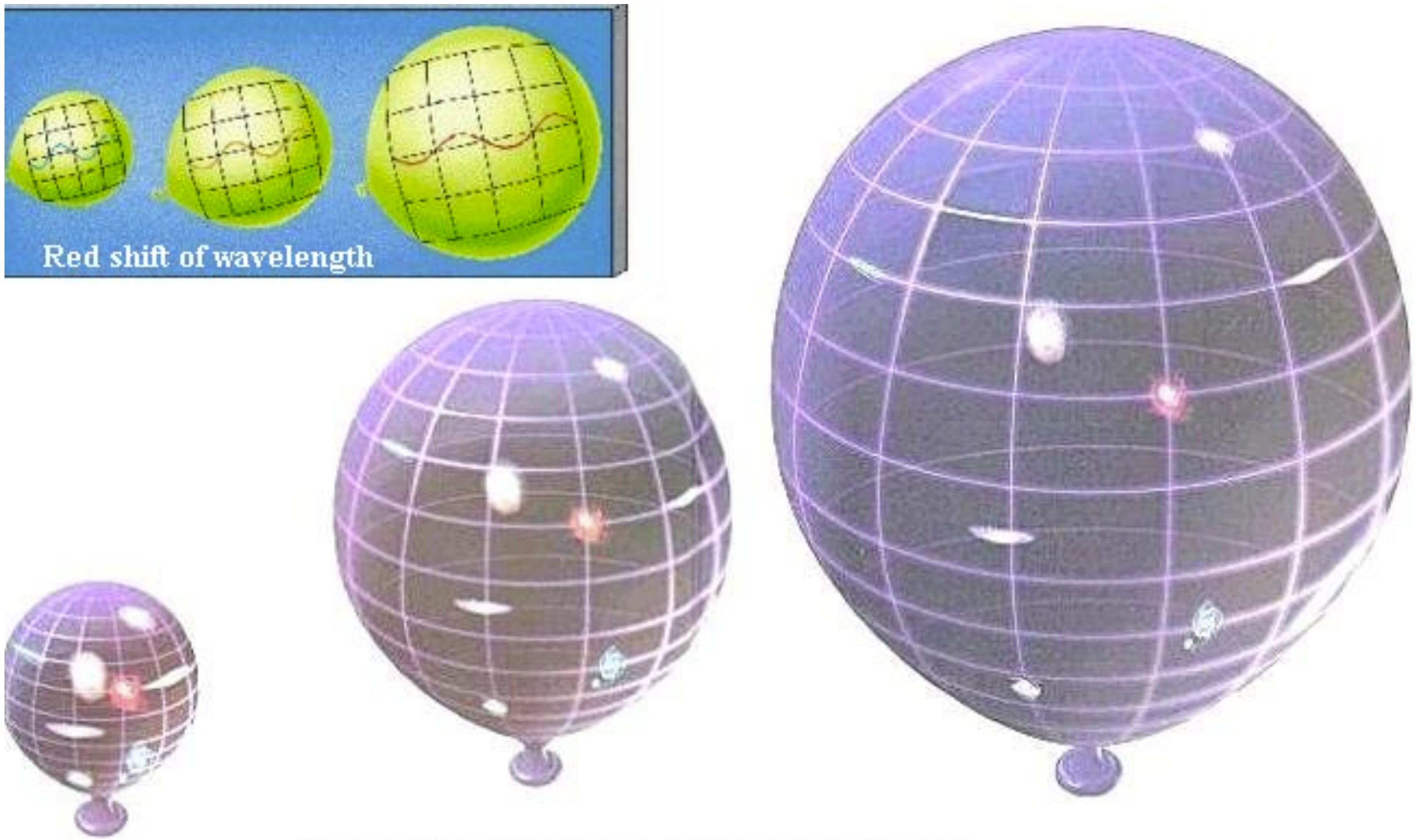
Alexander Friedmann (1888-1925)
Russian Physicist and Mathematician

Georges Lemaître (1894-1966)
Belgian Catholic Priest and Astronomer

Howard P. Robertson (1903-1961)
USA physicist and mathematician

Arthur G. Walker (1909-2001)
UK Mathematician

Comoving Coordinates



Einstein/Ricci Tensor for FLRW metric

Non-zero components of the FLRW metric tensor

$$g_{00} = 1$$
$$g_{11} = -\frac{a^2(t)r^2}{1-kr^2}$$
$$g_{22} = -a^2(t)r^2$$
$$g_{33} = -a^2(t)r^2\sin^2\theta$$

Non-zero components of the FLRW Christoffer symbols

$$\Gamma^0_{11} = \frac{\dot{a}a}{1-kr^2} \quad ; \quad \Gamma^0_{22} = a\dot{a}r^2 \quad ; \quad \Gamma^0_{33} = a\dot{a}r^2\sin^2\theta$$
$$\Gamma^1_{01} = \Gamma^1_{10} = \Gamma^2_{02} = \Gamma^2_{20} = \Gamma^3_{03} = \Gamma^3_{30} = \frac{\dot{a}}{a}$$
$$\Gamma^1_{22} = -r(1-kr^2) \quad ; \quad \Gamma^1_{33} = -r(1-kr^2)\sin^2\theta$$
$$\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}$$
$$\Gamma^2_{33} = -\sin\theta\cos\theta \quad ; \quad \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta \quad ,$$

Non-zero components of the Ricci tensor

$$R_{00} = -3\frac{\ddot{a}}{a}$$
$$R_{11} = \frac{a\ddot{a}+2\dot{a}^2+2k}{1-kr^2}$$
$$R_{22} = r^2(a\ddot{a}+2\dot{a}^2+2kr)$$
$$R_{33} = r^2(a\ddot{a}+2\dot{a}^2+2kr)\sin^2\theta$$

Ricci curvature scalar

$$R = \frac{6}{a^2}(a\ddot{a}+2\dot{a}^2+k)$$

Energy-Momentum Tensor

$$\boxed{R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu}} = \boxed{8\pi T_{\mu\nu}}$$

Calculated in the previous slide

Still missing…but we can use again the cosmological principle:

E/m tensor for an isotropic homogeneous "fluid":

$$T_{\mu\nu} = (\rho + P)v_\mu v_\nu - P g_{\mu\nu}$$

Density

Pressure

4-velocity

Note this!

GR is in general "hard" to solve: the metric shows up on BOTH sides of the Einstein Equations..

The "fluid" is at rest wrt the co-moving coordinates: v=(1,0,0,0)

The Friedmann Equations

Putting together the previous calculations, we can obtain the Friedmann equations:

Acceleration equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)$$

Hubble par. equation:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$

The Friedmann equations are the Einstein equations for the FLRW metric and an isotropic homogenous fluid.

How to solve them for the scale factor a(t)? We need to know pressure and density of the “fluid”, or at least a relation between then: an equation of state.

Equation of State

Energy-momentum conservation: $\nabla_\mu T^\mu_\nu = \partial_\mu T^\mu_\nu + \Gamma^\mu_{\mu\beta} T^\beta_\nu - \Gamma^\beta_{\mu\nu} T^\mu_\beta = 0$

For the 0-th component: $\partial_0 \rho(t) + 3\frac{\dot{a}(t)}{a(t)}(\rho(t) + P(t)) = 0$

Choose the generic equation of state: $P = w\rho$

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a}$$

$$\rho(t) \propto a(t)^{-3(1+w)}$$

Cosmological Parameters

$$\boxed{\begin{aligned} H(t) &= \frac{1}{a}\frac{da}{dt} \quad \text{Hubble Parameter} \\ q(t) &= -\frac{1}{a}\frac{d^2a}{dt^2} / \left(\frac{1}{a}\frac{da}{dt}\right)^2 \quad \text{Deceleration} \end{aligned}}$$

$$j(t) = \frac{1}{a}\frac{d^3a}{dt^3} / \left(\frac{1}{a}\frac{da}{dt}\right)^3 \quad \text{"Jerk"}$$

$$s(t) = \frac{1}{a}\frac{d^4a}{dt^4} / \left(\frac{1}{a}\frac{da}{dt}\right)^4 \quad \text{"Snap" (or Jounce)}$$

$$c(t) = \frac{1}{a}\frac{d^5a}{dt^5} / \left(\frac{1}{a}\frac{da}{dt}\right)^5 \quad \text{"Crackle"}$$

$$p(t) = \frac{1}{a}\frac{d^6a}{dt^6} / \left(\frac{1}{a}\frac{da}{dt}\right)^6 \quad \text{"Pop"}$$

...

Hubble's Law

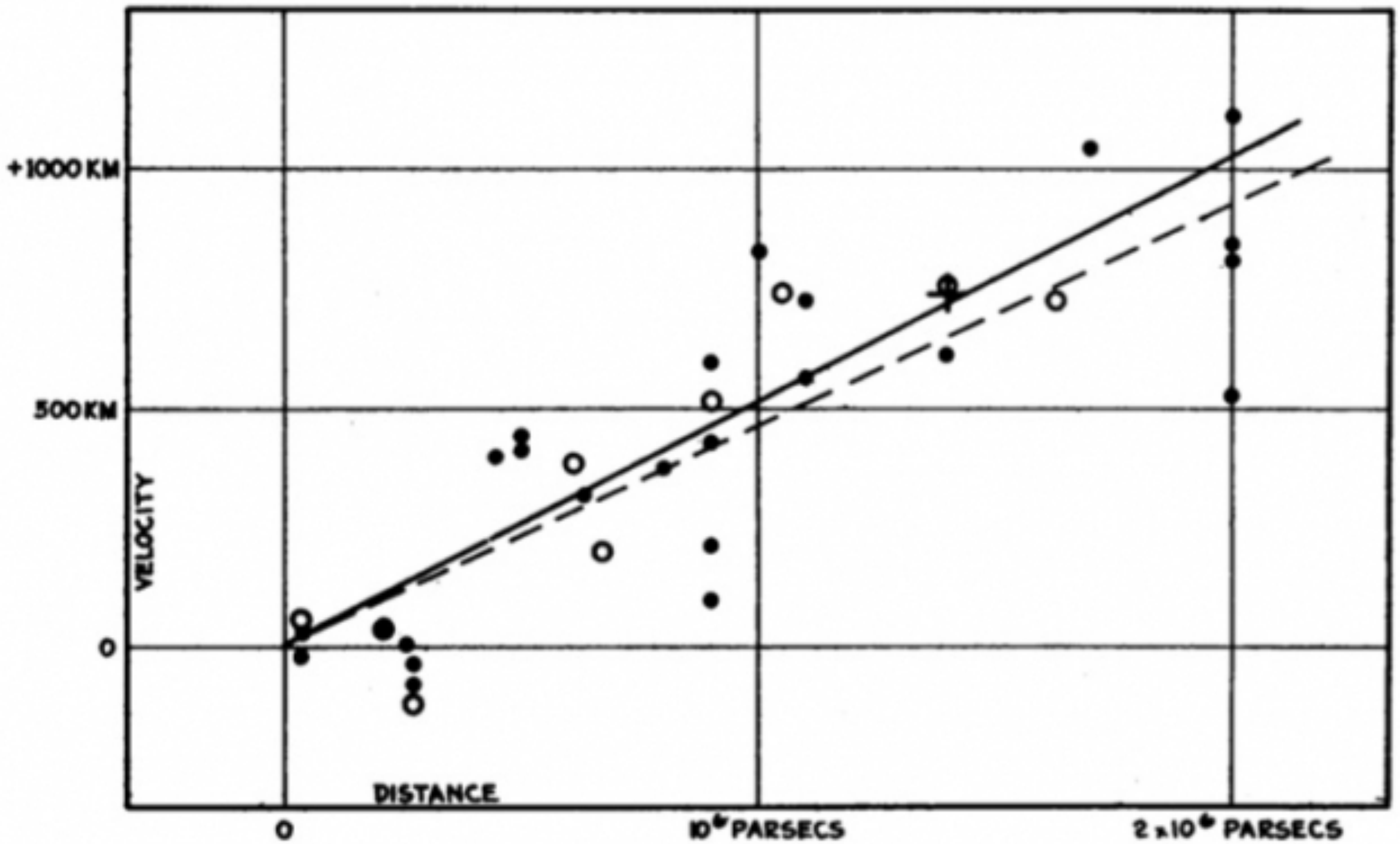
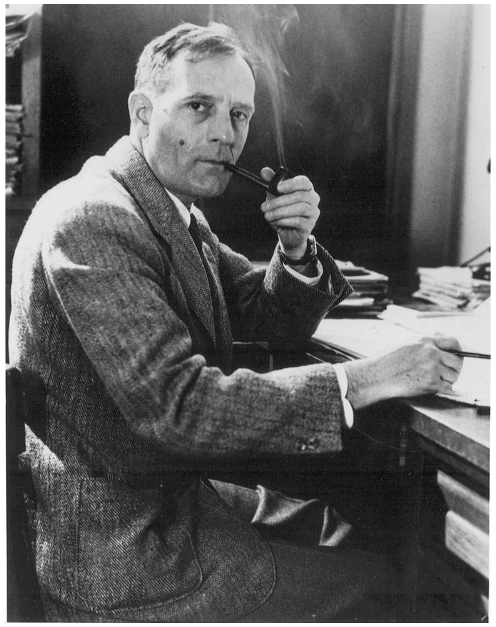


FIGURE 1
Velocity-Distance Relation among Extra-Galactic Nebulae.



Edwin Hubble (1889 – 1953)

$$d \propto v \quad \Rightarrow \quad \frac{\dot{a}}{a} = H \quad \Rightarrow \quad \dot{a} = Ha$$

Simplest Cosmological Models

Friedmann Equations
with Cosmological Constant Term

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = -\frac{4\pi G}{3}(\rho + 3P) + \boxed{\frac{\Lambda}{3}}$$

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \boxed{\frac{\Lambda}{3}} \; .$$

Equations of state

$$P = w\rho$$

$$w = 0 \Rightarrow \rho \sim \frac{1}{a^3} \quad \text{"Dust"}$$

$$w = 1/3 \Rightarrow \rho \sim \frac{1}{a^4} \quad \text{"Radiation"}$$

$$w = -1 \Rightarrow \rho \sim const. \quad \text{"Vacuum"}$$

Matter-Dominated Universe

"Dust": $w = 0 \quad \Rightarrow \quad \rho \cdot a(t)^3 = A$

Define the Conformal Time $\dfrac{d\eta}{dt} = \dfrac{1}{a}$

The 2nd Friedmann equation becomes $a'^2 = \dfrac{8\pi G}{3} A a - k a^2$

With solutions

$$
\begin{aligned}
k = 1 \Rightarrow a &= \frac{4\pi G A}{3}(1 - \cos\eta) \quad &;\quad t &= \frac{4\pi G A}{3}(\eta - \sin\eta) \\
k = 0 \Rightarrow a &= \frac{2\pi G A}{3}\eta^2 \quad &;\quad t &= \frac{2\pi G A}{9}\eta^3 \\
k = -1 \Rightarrow a &= \frac{4\pi G A}{3}(\cosh\eta - 1) \quad &;\quad t &= \frac{4\pi G A}{3}(\sinh\eta - \eta)
\end{aligned}
$$

Radiation-Dominated Universe

Radiation: $w = 1/3 \Rightarrow \quad \dot{a}^2 = \frac{8\pi G}{3}\frac{A}{a^2} - k$

With solutions

$$k = 1 \Rightarrow a = \sqrt{2\sqrt{\frac{8\pi GA}{3a^2}}t - t^2}$$

$$k = 0 \Rightarrow a = \sqrt{2\sqrt{\frac{8\pi GA}{3a^2}}t}$$

$$k = -1 \Rightarrow a = \sqrt{2\sqrt{\frac{8\pi GA}{3a^2}}t + t^2}$$

Vacuum-Dominated Universe

Vacuum: $$P = \rho = 0 \Rightarrow \;\; \dot{a}^2 = \frac{\Lambda a^2}{3} - k$$

With solutions

$$k = 1 \Rightarrow a = \sqrt{\frac{3}{\Lambda}} \cosh\left(\sqrt{\frac{\Lambda}{3}} t\right)$$

$$k = 0 \Rightarrow a = \sqrt{\frac{3}{\Lambda}} \exp\left(\sqrt{\frac{\Lambda}{3}} t\right)$$ ⟵ **Note this result**

$$k = -1 \Rightarrow a = \sqrt{\frac{3}{\Lambda}} \sinh\left(\sqrt{\frac{\Lambda}{3}} t\right)$$

The Density Parameter

Remember the Hubble parameter: $$H(t) = \frac{\dot{a}(t)}{a(t)}$$

The 2nd Friedmann Equation becomes $$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \Rightarrow \frac{8\pi G}{3H^2}\rho - 1 = \frac{k}{H^2 a^2}$$

Introducing the Critical Density $$\rho_c = \frac{3H^2}{8\pi G}$$

And the Density Parameter $\Omega = \rho/\rho_c$ we have: $$\boxed{\Omega - 1 = \frac{k}{H^2 a^2}}$$

and therefore

$$\begin{aligned} \rho < \rho_c &\Longleftrightarrow \Omega < 1 \Longleftrightarrow k = -1 && \text{(Open)} \\ \rho = \rho_c &\Longleftrightarrow \Omega = 1 \Longleftrightarrow k = 0 && \text{(Flat)} \\ \rho > \rho_c &\Longleftrightarrow \Omega > 1 \Longleftrightarrow k = 1 && \text{(Closed)} \end{aligned}$$

Mixed Models

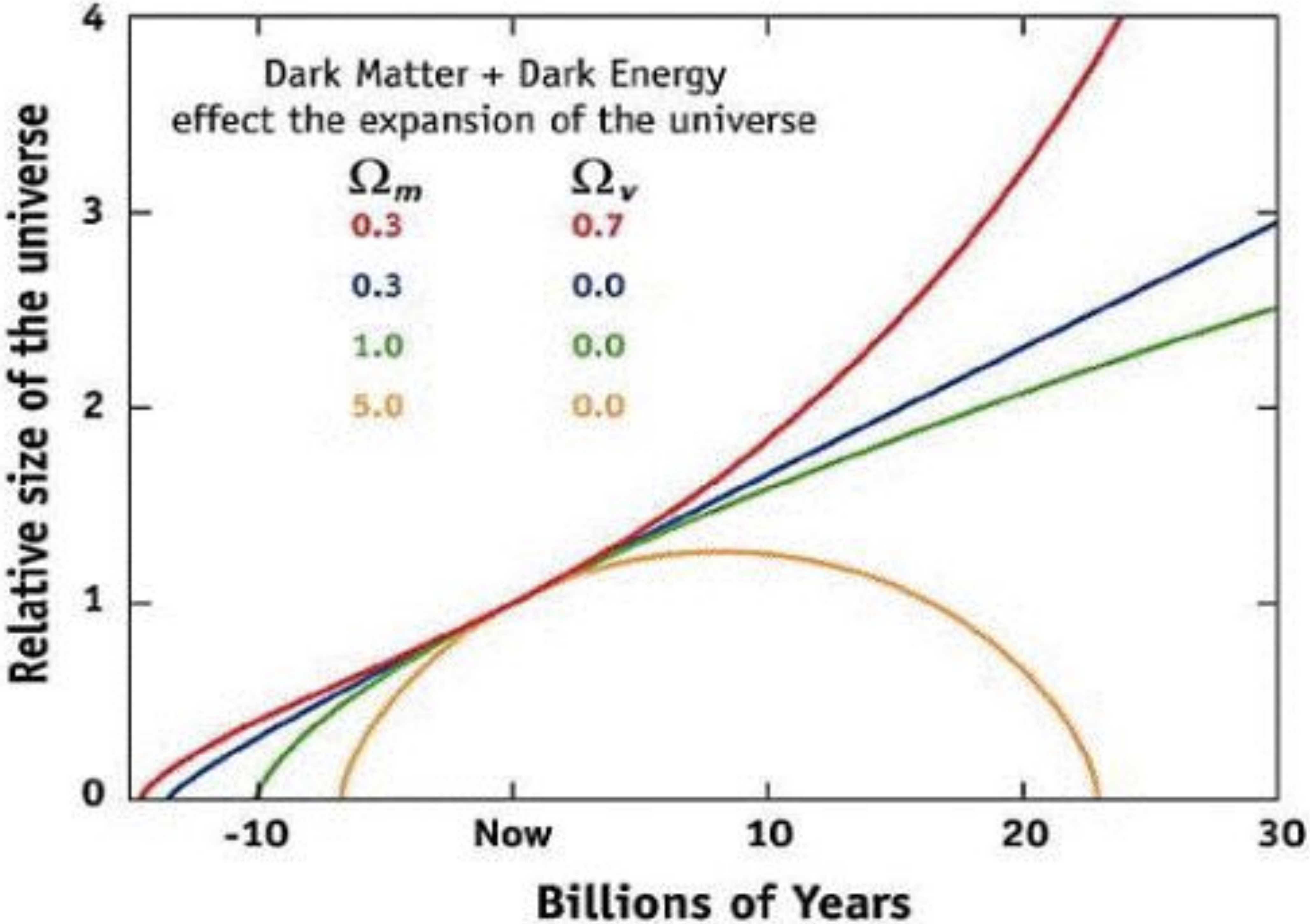
If more than a component is present $\rho_{TOT}(a) = \sum_i \rho_i(a) = \rho_C \sum_i \Omega_i a^{-3(1+w_1)}$

The Friedmann equation is then $\frac{k}{a^2} = H^2(\Omega_{TOT} - 1)$

And introducing the today's observed density parameters

$$\frac{k}{a_0^2} = H_0(\Omega_m + \Omega_r + \Omega_\Lambda - 1)$$

Summary (1)



Dark Matter + Dark Energy
effect the expansion of the universe
Ω_m
Ω_v
0.3 0.7
0.3 0.0
1.0 0.0
5.0 0.0
Relative size of the universe
0
1
2
3
4
-10
Now
10
20
30
Billions of Years

JG|U

Summary 2